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Geometry and Topology of

SOLITONS

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Contents

Abstract

Basic concepts and definitions in differential geometry and topology which are important in the theory of solitons and instantons are reviewed. Many examples from soliton theory are discussed briefly, in order to highlight the application of various geometrical concepts and techniques.

1. Introduction

Recent developments in soliton theory have been associated with frequent application of geometrical and topological ideas which provide an elegant interpretation of many soliton properties. The mathematical methods of differential geometry and topology are very abstract and rigorous which make them hard to grasp by a physics-oriented scientist. In this writing, I have tried to provide an interdisciplinary and informal introduction to those topics in differential geometry and topology which have proven important in soliton theory. A background knowledge of tensor calculus and field theory is assumed throughout this review.

2. Forms, fibers, and bundles

An $n-$**dimensional** *manifold* **is a space which behaves locally like** R^n**. In a similar manner, complex manifolds can be defined which are locally similar to** C^n**. A circle is a simple example of a one-dimensional manifold while a figure like** $+$ **is not a manifold because of its behavior at the junction point.** *Compact manifolds* **have a finite volume. An immediate example is the** $n-$**dimensional sphere** S^n **(** S^o **which contains only two points can be included in this definition). In contrast,** R^n **is an example of a noncompact manifold.**

Group manifolds **are spaces constructed by the free parameters which specify the elements of a group. For example, the group manifold of** Z_2 **is** S^o **while that of** $U(1)$ **is** S^1**, that of** $SU(2)$ **is** S^3**, that of** $SO(3)$ **is** $SU(2)/Z_2$ **or** $P_3(R)$**, etc.**

Consider a real vector E **space on the** $n-$**dimensional manifold** $\mathcal{M}$ **(we are mainly concerned with** R^n **and the Minkowski spacetime).** *1-forms* **are linear mappings from** E **to** R**:**

$$\omega(\alpha u+\beta v)=\alpha\omega(u)+\beta\omega(v) \tag{1}$$

where ω **is a 1-form ,** $\alpha,\beta\in R$**, and** $u,v\in E$**. A familiar 1-form in classical mechanics is the work 1-form** $\vec{F}.d\vec{x}$**.** $p-$*forms* $\omega(u_1,...,u_p)$ **are** p**-linear antisymmetric mappings from** p**-vectors** $E\times...\times E$ **(** p**-times) to** R**:**

$$\omega(u_1,...,\alpha u'_j+\beta u''_j,...,u_p)=\alpha\omega(u_1,...,u'_j,...,u_p)+\beta\omega(u_1,...,u''_j,...,u_p), \tag{2}$$

with

$$\omega(u_{i_1},...,u_{i_p})=sgn(\pi)\omega(u_1,...,u_p) \tag{3}$$

where $sgn(\pi)$ **=+1(-1) if the permutation** $\pi:(1,...,p)\rightarrow(i_1,...,i_p)$ **is even (odd). Consider the** $p-$**form** ω **and the** $q-$**form** η **such that** $p+q$ **is less than or equal to the dimension of the vector space** E**. The** *exterior or wedge product* $\omega\wedge\eta$ **is a** $(p+q)-$**form such that**

$$\eta\wedge\omega=(-1)^{pq}\omega\wedge\eta \tag{4}$$

This product is distributive with respect to addition, and associative:

$$\omega \wedge (\eta + \zeta) = \omega \wedge \eta + \omega \wedge \zeta,$$

and

$$\omega \wedge (\eta \wedge \zeta) = (\omega \wedge \eta) \wedge \zeta. \tag{5}$$

The space of all tangent vectors to a manifold $\mathcal{M}$ at the point x is called the *tangent space* **and is denoted by $T_x\mathcal{M}$. This space has the same dimension n as the manifold $\mathcal{M}$. The union of all such tangent spaces form the** *tangent bundle* **$T\mathcal{M}$ and is a manifold of dimension $2n$.**

While $\{\frac{\partial}{\partial x^i}\}$ forms a basis of the tangent space $T_x\mathcal{M}$, the dual basis $\{dx^i\}$ forms a basis of the so-called *cotangent space* **$T^*_x(\mathcal{M})$. The inner product of these two bases satisfy**

$$(\frac{\partial}{\partial x^i}|dx^j) = \delta^j_i \tag{6}$$

Differential $p-$forms **are $p-$forms on the tangent space $T\mathcal{M}$. Consider, for example, a real function $F(x, y, z)$ on the three dimensional Euclidean space R^3. F is in fact a 0-form, while $dF = \partial_i F dx^i$ is a differential 1-form, similar to the work form. The flux 2-form**

$$\Phi = \Phi_x dy \wedge dz + \Phi_y dz \wedge dx + \Phi_z dx \wedge dy \tag{7}$$

is another example of a differential form. In Minkowski spacetime, the electromagnetic potentials form a 1-form A with components A_μ ($\mu = 0, 1, 2, 3$). The components of a $p-$form coincide with those of an anti-symmetric covariant tensor of rank p for $p > 1$. For $p = 0$, the 0-form transforms like a scalar and $p = 1$ forms correspond to covariant vectors.

The tensor product $\omega \otimes \eta$ of the $p-$form ω with the $q-$form η does not make a $(p + q)-$form since $\omega \otimes \eta$ is not antisymmetric with respect to all of its components. The wedge product $\omega \wedge \eta$ is defined in such a way to preserve the antisymmetry. For example, the wedge product of two 1-forms η and ω satisfies $\omega \wedge \eta = \omega \otimes \eta - \eta \otimes \omega$. 1-forms, therefore, anticommute ($\omega \wedge \eta = -\eta \wedge \omega$).

The exterior derivative of a $p-$form ω is defined in such a way to lead to a $(p+1)$-form. Consider, for example, the 1-form A on M:

$$A = A_o dt + A_1 dx + A_2 dy + A_3 dz = A_\mu dx^\mu. \tag{8}$$

We have

$$\begin{aligned} dA = dA_o \wedge dt + dA_1 \wedge dx + dA_2 \wedge dy + dA_3 \wedge dz = dA_\mu \wedge dx^\mu \\ = (\partial_\alpha A_\mu dx^\alpha) \wedge dx^\mu = \partial_\alpha A_\mu dx^\alpha \wedge dx^\mu \\ = (\partial_y A_3 - \partial_z A_2) dy \wedge dz + ... \end{aligned} \tag{9}$$

The *Hodge operation* *** on a $p-$form ω produces an $(n - p)$-form $^*\omega$ according to**

$$(*\omega)_{ij...} = \frac{1}{p!} \epsilon_{lm...ij...} \omega^{lm...} \tag{10}$$

where $\epsilon_{lm...}$ (n indices) is the totally antisymmetric tensor of the $n-$dimensional space. Note that * is defined in terms of components (the components of $p-$forms have p indices). The antisymmetry of differential forms implies

$$dd\omega = 0 \tag{11}$$

for any $p-$form ω. It can also be shown that

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta \tag{12}$$

for any $p-$form ω and $q-$form η. The $p-$form ω is called *closed* **(globally) if $d\omega = 0$ and** *exact* **if $\omega = d\eta$ where η is a $(p-1)$-form. Note that all exact forms are closed, but the converse is not always true. The reader recalls that not all vector fields can be written as the gradients of scalar functions.**

The electromagnetic field 2-form F defined by

$$F = dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{13}$$

is an example of an exact form. The action for the free Maxwell field can be written in the form

$$-\int \frac{1}{2}\, {}^*dA \wedge dA. \tag{14}$$

Note that if $F_{\alpha\beta}$ are the components of a two-form F, $(dF)_{\alpha\beta\gamma} = \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}$, which is antisymmetric with respect to all indices. It can also be shown that if ω is closed and η is exact, $\omega \wedge \eta$ will be exact. The reader can verify these two statements as exercises.

The well-known identities $\vec{\nabla} \times \vec{\nabla}\phi = 0$ and $\vec{\nabla}.\vec{\nabla} \times \vec{F} = 0$ in vector analysis follow from the generalized identity $dd = 0$. Consider, for example, the 1-form

$$\omega = u_i dx^i \tag{15}$$

in the three dimensional Euclidean space where u_i ($i = 1, 2, 3$) are functions of x^i ($x^1 = x,\ x^2 = y,\ x^3 = z$). The exterior derivative of this 1-form yields $\vec{\nabla} \times \vec{u}$:

$$d\omega = d(u_i dx^i) = \frac{\partial u_i}{\partial x^j} dx^j \wedge dx^i$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i}\right) dx^j \wedge dx^i = \frac{1}{2} \epsilon_{kji} (\vec{\nabla} \times \vec{u})_k dx^j \wedge dx^i \tag{16}$$

and

$$dd\omega = \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial x^k \partial x^j} - \frac{\partial^2 u_j}{\partial x^k \partial x^i}\right) dx^k \wedge dx^j \wedge dx^i = 0 \tag{17}$$

The last identity follows from the symmetry of $\frac{\partial^2}{\partial x^k \partial x^j}$ and $\frac{\partial^2}{\partial x^k \partial x^i}$ and antisymmetry of $dx^k \wedge dx^j$ and $dx^k \wedge dx^i$.

A $p-$form which can be expressed as $\omega = d\phi_1 \wedge ... \wedge d\phi_p$ where $\phi_1, ...\phi_p$ are scalar fields, is called *simple*. A simple form is both exact and closed. Note that ω can be written as $\omega = d(\phi_1 d\phi_2 \wedge ... \wedge d\phi_p)$.

The *co-differential* of the $p-$form ω is defined as

$$\delta\omega = (-1)^{p(n-p+1)} \; {}^*d^*\omega \tag{18}$$

for Euclidean metrics. The definition for Minkowskian metrics differ by a - sign. Note that if ω is a $p-$form, $\delta\omega$ is a $(p-1)-$form. Also $\delta\delta = \pm \; {}^*d^{2*} = 0$. The Laplacian operator is obtained by forming $\triangle = -(\delta d + d\delta)$, which in a Minkowski spacetime becomes the d'Alembertian operator. A differential form ω is called *harmonic* if $\triangle\omega = 0$. The inhomogeneous Maxwell equations can also be written in the form $\delta F = J$ and the conservation of electric current as $\delta J = 0$.

Integration of $p-$forms over the space of interest (or part of it) is of great importance. In R^n, $dx^1 \wedge ... \wedge dx^n = dx^1 ... dx^n$ provides the volume $n-$form. For a general metric, the volume form is given by $\sqrt{|g|}dx^1...dx^n$, where g is the determinant of the metric tensor. *Stokes's theorem* reads

$$\int_V d\omega = \int_{\partial V} \omega \tag{19}$$

where the LHS integration is over the submanifold V with the boundary ∂V. The manifold over which the integration is performed (and hence its boundary) is assumed to be *orientable*. The existence of a volume form on the manifold of interest guarantees its orientability. *Klein bottle* and *Möbius strip* are examples of nonorientable manifolds. Stokes's theorem leads to the more common Stokes and divergence theorems in vector analysis.

Consider a 3-dimensional spacelike volume V with the boundary ∂V. The magnetic flux through ∂V is given by $\int_{\partial V} F$, while the electric flux is given by $\int_{\partial V} \; {}^*F$ which vanishes in ordinary electromagnetism. The electric charge contained in V is given by $-\int_V \; {}^*J$. The Gauss theorem can therefore be written as

$$\int_{\partial V} \; {}^*F = -\int_V \; {}^*J.$$

The following results are relevant to our discussion:

- Any $k-$dimensional regular domain X of a manifold $\mathcal{M}$ (see section 3) has a boundary ∂X which is a $(k-1)$-dimensional compact manifold. ∂X itself has no boundary, i.e. $\partial\partial X = 0$. This is called *Cartan's lemma* .
- If $\mathcal{M}$ is a simply connected manifold (see section 9), then $\oint_C \omega = 0$ for an arbitrary closed 1-form ω. C is a closed curve in $\mathcal{M}$.
- If $\mathcal{M}$ is simply connected, then all closed 1-forms ω on $\mathcal{M}$ are exact.

- **Consider a compact, oriented, $p-$dimensional submanifold X of the simply connected manifold $\mathcal{M}$. Then, for any two cohomologous (see section 8) $p-$forms ω and η on $\mathcal{M}$, $\int_X \omega = \int_X \eta$.**
- **A closed $p-$form ω on S^p is exact if and only if $\int_{S^p} \omega = 0$.**

Note that any singularities in the fields described by differential forms must be treated as holes in the manifold on which the forms are defined. For example, consider $d\theta$ on the xy-plane where $\theta = arctan(y/x)$:

$$d\theta = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \tag{20}$$

This form is closed but not exact. While every closed $p-$form ($p > 0$) on R^n is exact, (20) is not exact because of the singularity at $x = y = 0$ which introduces a hole in R^2. In other words, the manifold on which (20) is defined is $R^2 - \{0\}$. Any closed 1-form ω on $R^2 - \{0\}$ integrates on a smooth closed curve C according to

$$\oint_C \omega = w(C, 0) \int_{S^1} \omega \tag{21}$$

where $w(C, 0)$ is the winding number of C with respect to the origin (see section 3).

The *Hilbert product* of forms is defined as follows

$$< \omega|\eta >= \int_{\mathcal{M}} {}^*\omega \wedge \eta = \int_{\mathcal{M}} \frac{1}{p!}\omega^{i_1...i_p}\eta_{i_1...i_p}\sqrt{|g|}dx^1...dx^n \tag{22}$$

where g is the determinant of the manifold's metric. Note that ω and η are both k-forms. For compact manifolds the Hilbert product is always well-defined. It can be easily shown that $< {}^*\omega| {}^*\eta >= \pm < \omega|\eta >$ depending on the signature being Euclidean (+) or Minkowskian (-).

As an example, consider the action for a real Klein-Gordon field

$$S = \int (\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2)\sqrt{|g|}d^4x$$

which can be written as

$$S = \frac{1}{2} < d\phi|d\phi > -\frac{1}{2}m^2 < \phi|\phi >= \int (\frac{1}{2} {}^*d\phi \wedge d\phi - \frac{1}{2}m^2 {}^*\phi \wedge \phi). \tag{23}$$

The tangent space $T_x\mathcal{M}$ is the prototype of a *fiber*. In order to define a fiber more generally, consider the triplet (T, B, π), where T is the *bundle space* (or *total space*), B is the base space and π is a C^∞ (i.e. infinitely differentiable) mapping from T to B. Such a triplet is called a *fibration*. An example is $(\mathcal{M} \times V, \mathcal{M}, \pi)$ where $\mathcal{M} \times V$ is the product manifold of $\mathcal{M}$ with an arbitrary $m-$dimensional vector space V. This is called a *local vector bundle of rank m* . The projection map π is simply $\pi(x, u) = x$ in this case, where $x \in \mathcal{M}$ and $u \in V$. Part of the total space which sits on top of the point $x \in B$ is $\pi^{-1}(x)$ and is called a *fiber*. The manifold T can

therefore be considered as a collection of fibers or a *fiber bundle*. A C^∞ mapping $f : B \to T$ such that $\pi \circ f = Id$ is called a *section* of the fibration. Here, Id_B is the identity map on B ($Id_B(x) = x$).

A fiber bundle is *locally trivializable* if it can be described by the product manifold $U_i \times F$ where U_i is a neighborhood of the base manifold and F is the fiber. Since local properties are not sufficient to describe the global topology of the bundle, a set of *transition functions* ϕ_{ij} are defined which specify how the fibers are related to each other in the overlapping region of the two neighborhoods U_i and U_j. The transition function ϕ_{ij} is therefore defined as the mapping of the fibers on U_i to the fibers on U_j over the region $U_i \cap U_j$. For a trivial fiber bundle, all the transition functions can be the identity map. As we said before, $\{\frac{\partial}{\partial x^\mu}\}$ is the standard basis for the local frames of the tangent bundle $T\mathcal{M}$ while $\{dx^\mu\}$ is the basis for the cotangent bundle $T^*\mathcal{M}$. Since $\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial}{\partial x'^\alpha}$ and $dx'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha}dx^\alpha$, they define transition functions between two overlapping neighborhoods U and U' of the tangent and cotangent bundles, respectively.

In gauge field theories, we encounter a special type of fibration which is called a *principal fibration*. A principal fibration consists of a base manifold $\mathcal{M}$, the structure group G, and the manifold P on which the Lie group G acts. Like in an ordinary fibration, π is a mapping from P to $\mathcal{M}$. The principal fibration, therefore, is denoted by $(P, G, \mathcal{M}, \pi)$. If G is a gauge group, a section in the principal fibration corresponds to the choice of a particular gauge. In the case that there exists a global C^∞ section, the principal fibration is said to be *trivializable* . A trivializable fibration is isomorphic to $(\mathcal{M} \times G, G, \mathcal{M}, pr_1)$.

Any fiber bundle is trivial if the base space is contractible. Non-trivial fiber bundles can therefore exist over topologically non-trivial base manifolds (like $R^3 - 0$, S^1, etc.). Only for a trivial principal bundle can one find a single gauge potential which is smooth over the entire base manifold.

The fiber of a vector bundle is a linear vector space. The transition functions of a vector bundle are elements of the *general linear group* of the vector space. Similarly, the transition functions of a principal bundle are elements of G acting by left multiplication. The associated vector bundle $PX_\rho V$ is defined using the representation $R(G)$ acting on the finite-dimensional vector space V. The *associated vector bundle* is based on the equivalence relation

$$(p, \rho(g) \circ v) \simeq (p \circ g, v) \quad \forall \quad p \in P, \quad v \in V, \quad g \in G, \quad and \quad \rho \in R(G). \tag{24}$$

The tangent space to the bundle space T_pP can be decomposed into *vertical* and *horizontal* parts:

$$T_pP = H_p \oplus V_p \tag{25}$$

The vertical part which corresponds to the action of G maps into a single

point of the base space. The horizontal part leads to the *connection* **1-form which will be defined in the next section.**

The space of spinors is also a vector space and *spinor bundles* **can be constructed in a similar way as the vector bundles. The corresponding principal bundle has the spin group with the** *Clifford algebra* **as the tangent space at the identity element.**

3. Smooth maps and winding numbers

Consider a smooth map from the $n-$dimensional differentiable manifold $\mathcal{M}$ to the $m-$dimensional differentiable manifold $\mathcal{M}'$

$$f : \mathcal{M} \rightarrow \mathcal{M}' \tag{26}$$

This mapping is called

- *surjective*, **if $f(\mathcal{M}) = \mathcal{M}'$. In other words, for all $y \in \mathcal{M}'$ there is at least one element $x \in \mathcal{M}$ such that $f(x) = y$.**
- *injective*, **if for all $x_1, x_2 \in \mathcal{M}$ where $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$. In other words, distinct points in $\mathcal{M}$ have distinct images.**
- *bijective*, **if it is both surjective and injective. The existence of a bijective mapping from $\mathcal{M}$ to $\mathcal{M}'$ ensures that the points in these two spaces are in one-to-one correspondence.**

For details, the reader is referred to Long (1971).

Let x^i ($i = 1, ..., n$) and y^j ($j = 1, ..., m$) be coordinate systems in $\mathcal{M}$ and $\mathcal{M}'$, respectively. These coordinates are related by the map f. We can expand $y^j(x^i)$ around a point p in $\mathcal{M}$ with coordinates x^i_o

$$y^j(x^i_o + \Delta x^i) = y^j(x^i_o) + \left(\frac{\partial y^j}{\partial x^i}\right)_p \Delta x^i + ... \tag{27}$$

Note that $[\frac{\partial y^j}{\partial x^i}]$ is an $m \times n$ matrix called the *Jacobi matrix*. **This matrix defines the appropriate linear mapping from $T_p(\mathcal{M})$ to $T_{f(p)}(\mathcal{M}')$ (see Felsager, 1983).**

A well-defined transformation between the two tangent spaces requires the mapping to be a *diffeomorphism*, **which in simple terms means that the map is bijective with an inverse f^{-1} which is smooth. For $m = n$, the Jacobi matrix becomes a non-singular square matrix. If $m < n$ and f is everywhere regular, the map is called an** *immersion*, **and $f(\mathcal{M})$ is a submanifold of $\mathcal{M}'$. An** *embedding* **is an immersion which is further required to be a** *homeomorphism*. **If $m > n$, and f is everywhere regular, it is called a** *submersion*. **We now assume that $\mathcal{M}$ and $\mathcal{M}'$ are both compact and orientable, and have the same dimension n. The** *Brouwer degree* **of the map f is defined as**

$$deg(f) = \sum_{p_i} sgn|\frac{\partial y^j}{\partial x^i}|_{p_i} \tag{28}$$

where $f(p_i)$ are the regular values in $\mathcal{M}'$. A point $x \in \mathcal{M}$ is said to be *regular* if $f'(x) \neq 0$, otherwise, it is a *critical point*. A point $\xi \in \mathcal{M}'$ is either the image of a regular point, the image of a critical point, or it is not the image of any point. To see what a regular value means, consider a map $f : S^1 \to S^1$. In this example, the map $\xi = \theta$ is of degree $+1$, $\xi = -\theta$ is of degree -1, and the map $\xi = \frac{1}{\pi}(\theta - \pi)^2$ has a vanishing degree. Note also that $\xi = n\theta$ is of degree n.

The Brouwer degree of a map measures the net number of times $\mathcal{M}'$ is covered when all the points in $\mathcal{M}$ are swept once. The integer $deg(f)$ is also called *winding number*. The Brouwer degree vanishes for maps which are not surjective.

Brouwer's theorem states that

$$\int_{\mathcal{M}} f^*\omega = deg(f) \int_{\mathcal{M}'} \omega \tag{29}$$

where ω is a $p-$form on $\mathcal{M}'$ and $f^*\omega$ is its pullback, which is a $p-$form on $\mathcal{M}$ and will be defined in the next section.

4. Gauge Fields as connections on principal bundles

Consider a gauge group represented by $m \times m$ complex matrices acting on a complex $m-$dimensional vector space V. A vector bundle can be constructed with fibers isomorphic to V. A connection on this vector bundle is a 1-form on the base space with values in $C(m \times m)$ (the space of complex $m \times m$ matrices). The connection 1-forms of $SU(N)$, for example, are anti-hermitian, traceless $N \times N$ matrices and at the same time 1-forms on the base manifold (e.g. the Minkowski space). For a smooth section S of the principal bundle, the covariant derivative is defined according to

$$DS = ds + A \wedge S \tag{30}$$

where $A = A_\mu dx^\mu$ is the connection 1-form. Note that each A_μ is a complex $N \times N$ matrix, expandable in terms of the bases of $SU(N)$:

$$A_\mu = A^a_\mu T_a. \tag{31}$$

In the case of $SU(2)$, $T_a = \tau_a$ (a=1,2,3) are the Pauli matrices

$$\tau_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{32}$$

The Lie algebra of a group G is the tangent space to the group manifold at the identity element T_eG. The basis $\{T_a\}$ of this space obeys the algebra

$$[T_a, T_b] = f_{abc} T_c \tag{33}$$

where f_{abc} are the *structure constants* of the group.

Unlike the exterior derivative of ordinary 1-forms for which we have $ddS = 0$, DDS does not always vanish. This quantity which is a 2-form

leads in a natural way to the concept of curvature. Taking the covariant derivative of (31),

$$DDS = D(A \wedge S) = dA \wedge S + A \wedge DS = F \wedge S, \tag{34}$$

where the curvature 2-form is

$$F = dA + A \wedge A. \tag{35}$$

It can be shown that the covariant derivative of F vanishes

$$DF = dF + A \wedge F - F \wedge A = 0, \tag{36}$$

which is the generalized form of the *Bianchi identity*. Like A, F is also in the form of an $m \times m$ complex matrix. F is called curvature because it is related to the Gaussian curvature when a curved manifold is concerned. The Bianchi identity constitutes one of the basic equations governing the gauge field A. An element g of the gauge group linearly transforms a vector $v \in V$

$$v \rightarrow gv. \tag{37}$$

Such a transformation is associated with the following gauge transformations on the connection and curvature:

$$A \rightarrow Ad(g^{-1})A + g^{-1}dg, \tag{38}$$

and

$$F \rightarrow gFg^{-1}, \tag{39}$$

where $Ad(g^{-1})$ is the adjoint representation of g^{-1}. For a single point as the base space, the bundle space reduces to the Lie group G and the covariant derivative of the connection 1-form vanishes

$$dA + A \wedge A = 0. \tag{40}$$

This equation is called the *Maurer-Cartan structure equation*. The *Maurer-Cartan form* $g^{-1}dg$ belongs to the Lie algebra of the principal bundle. This form is invariant under the left action of a constant group element g_o:

$$(g')^{-1}dg' = (g_o g)^{-1}d(g_o g) = g^{-1}dg \tag{41}$$

This form can be parametrized as $g^{-1}dg = \phi_a T_a$ where T_a satisfy the algebra (34). Using the identity $d(g^{-1}dg) = -g^{-1}dg \wedge g^{-1}dg$ we obtain

$$d\phi_a + \frac{1}{2} f_{abc}\phi_a \wedge \phi_b = 0 \tag{42}$$

As we move along a curve $x^\mu(\lambda)$ in the base manifold, a corresponding section $S(\lambda)$ (called a *lift*) is traced in the principal bundle according to

$$\frac{d}{d\lambda} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{S}\frac{\partial}{\partial S} \tag{43}$$

The section S is said to be parallel transported if

$$\dot{S}_{ij} + A_{\mu,ik}\dot{x}^{\mu} S_{kj} = 0. \tag{44}$$

From (44) and (45) we obtain

$$\frac{d}{d\lambda} = \dot{x}^{\mu}\left(\frac{\partial}{\partial x^{\mu}} - A^{a}_{\mu} T^{a}_{ij} S_{jk} \frac{\partial}{\partial S_{ik}}\right). \tag{45}$$

The expression inside the parentheses gives the *covariant derivative* denoted by D_{μ}:

$$D_{\mu} = \partial_{\mu} - A^{a}_{\mu} T_{a} \tag{46}$$

where $T_a = T^{a}_{ij} S_{jk} \frac{\partial}{\partial S_{ik}}$. The basic motivation for defining the covariant derivative is to modify ∂_{μ} in such a way that the resulting quantity transforms covariantly under the action of the group element g. The components of the curvature 2-form ($F^{a}_{\mu\nu}$) are related to D_{μ} according to

$$[D_{\mu}, D_{\nu}] = -F^{a}_{\mu\nu} T_{a}, \tag{47}$$

or

$$F^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + f_{abc} A^{b}_{\mu} A^{c}_{\nu}. \tag{48}$$

In the language of differential forms,

$$\Omega = d\omega + \omega \wedge \omega = g^{-1} F g \tag{49}$$

where $\omega = g^{-1} A g + g^{-1} dg$ is the connection 1-form. Equations (36) and (49) can be combined to obtain

$$F = dA + A \wedge A = \frac{1}{2} F^{a}_{\mu\nu} T^{a} dx^{\mu} \wedge dx^{\nu}. \tag{50}$$

The inhomogeneous equations governing the gauge field read

$$D^{*}F = {}^{*}J \tag{51}$$

where J is the current 1-form. Conservation of the current J results from the underlying gauge symmetry (Noether's theorem)

$$D^{*}J = 0 \tag{52}$$

Note that it is the dual of the current 1-form which is covariantly closed. The source-free field equations can be obtained from the following action:

$$\mathcal{A} = \int \frac{1}{4} F \wedge^{*} F \tag{53}$$

Note that $F \wedge {}^{*}F$ is a 4-form proportional to the volume 4-form in the 4-dimensional Minkowski space.

In the case of the abelian $U(1)$ gauge symmetry, the group manifold is a circle parametrized by the angle θ ($g = e^{i\theta}$). The group element g acts on the vector space of complex functions ϕ. The principal bundle

is locally isomorphic to the product of an open set in the Minkowski space and S^1. The potential 1-form $A = A_\mu dx^\mu$ has real components A_μ in this case. We also have $F = dA = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$ with components as a covariant antisymmetric tensor

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{54}$$

Note that $A \wedge A = 0$, since in this case the potential 1-form is not matrix valued and $A_\mu A_\nu dx^\mu \wedge dx^\nu$ vanishes due to the symmetry of $A_\mu A_\nu$ and antisymmetry of $dx^\mu \wedge dx^\nu$. Also note that $A \wedge F = 0$ and the Bianchi identity (37) translates into the following tensorial relation

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{55}$$

while (52) becomes

$$\partial_\beta F^{\alpha\beta} = J^\alpha \tag{56}$$

Equations (56) and (57) constitute the complete Maxwell equations.

The $U(1)$ gauge transformation $\phi \rightarrow e^{i\theta}\phi$ leads to

$$A \rightarrow A + d\theta, \qquad or \qquad A_\mu \rightarrow A_\mu + \partial_\mu \theta \tag{57}$$

while F remains gauge invariant. A section of the principal bundle corresponds to the selection of a particular gauge $\theta(x^\mu)$.

Maxwell's equations imply the nonexistance of magnetic monopoles. The possibility of having magnetic monopoles and its implications has been extensively studied in the literature (see Goddard and Olive, 1978 and also Horva'thy, 1988 for an introductory exposition of the subject).

In the presence of magnetic charges, the homogeneous Maxwell equation is modified as

$$dF = -{}^*K \tag{58}$$

where K is the magnetic current 1-form. Expressed in tensor components, this equation reads

$$\frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|}\ {}^*F^{\mu\nu}) = K^\nu \tag{59}$$

or

$$\partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = -\sqrt{|g|} \epsilon_{\alpha\beta\gamma\delta} K^\delta \tag{60}$$

A point-like monopole located at the origin of spherical coordinates generates the following field

$$F = \frac{g}{4\pi} sin\theta d\theta \wedge d\phi \tag{61}$$

where g is the strength (magnetic charge) of the monopole. From $F = dA$ the potential 1-form can be chosen as

$$A = -\frac{g}{4\pi} cos\theta d\phi \tag{62}$$

The total magnetic flux across a sphere centered at the origin is

$$\Phi_B = \oint_{S^2} F = \frac{g}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} sin\theta d\theta d\phi = g \tag{63}$$

Note that the gauge potential (63) is *not* smooth everywhere. By computing the components of A in Cartesian coordinates, it is easily seen that the potential diverges along the z-axis. Expressed in mathematical terms, this means that the corresponding bundle is not trivializable and a global section does not exist.

The total angular momentum of the electromagnetic field produced by a static pair of a point charge q and a magnetic monopole g is given by

$$\vec{J} = \int \vec{x} \times (\epsilon_o \vec{E} \times \vec{B}) d^3x = \frac{qg}{4\pi}\hat{k} \tag{64}$$

where $\hat{k}$ is a unit vector from q to g. Using the quantum mechanical quantization $J = n\frac{\hbar}{2}$, we are led to an explanation for the quantization of the electric charge q in the presence of a magnetic monopole. The Dirac monopole will be considered again in section 8.

Transition functions on principal bundles play the role of gauge transformations. Two fiber coordinates ϕ and ϕ' in $U \cap U'$ transform to each other by $\phi' = g\phi$ where g is the transition function. Under this transformation,

$$A' = gAg^{-1} + gdg^{-1}, \quad \omega' = \omega, \quad F' = gFg^{-1}, \quad and \quad \Omega' = \Omega. \tag{65}$$

In mathematical terms, A and F are called *pullbacks* of ω and Ω, respectively.

The correspondence between the principal bundles and gauge fields can be summarized as follows

Structure group $\leftrightarrow$ Gauge group
Connection pullback (A) $\leftrightarrow$ Gauge potential
Curvature pullback (F) $\leftrightarrow$ Field strength
Associated vector bundles ψ $\leftrightarrow$ Matter fields
Transition functions $\leftrightarrow$ Gauge transformations
Sections $\leftrightarrow$ Gauge fixing conditions
Maurer-Cartan 1-forms $\leftrightarrow$ Pure gauges

The global topology of gauge fields become relevant in the path integral quantization approach. Path integral formalism works properly only for (++++)-signature spaces. This is why the Euclidean signature is of particular importance in the soliton and instanton methods.

For a detailed discussion of gauge theories and differential geometry, the reader is referred to Eguchi et al. (1980).

5. Yang-Mills field

$SU(2)$ gauge theory was introduced in 1954 by Yang and Mills. The gauge group of the Yang-Mills field is $SU(2)$. Elements of this group are unitary 2×2 matrices which operate on the two-component isospinors ψ. The covariant derivative D_μ is defined in such a way that $D_\mu \psi$ transform in the same way as ψ. According to equation (47)

$$D_\mu = \partial_\mu - A^a_\mu \tau_a \tag{66}$$

where τ_a are the Pauli matrices given in (33). Note that this is a matrix-valued equation and ∂_μ is implicitly assumed to be $I.\partial_\mu$ where I is the 2×2 unit matrix. The tangent space at the group identity $e = I$ defines the Lie algebra of the group. The algebra of group generators τ_a is

$$[\tau_a, \tau_b] = 2i\epsilon_{abc}\tau_c \tag{67}$$

where ϵ_{abc} is the totally antisymmetric tensor in three dimensions. Note that according to (48) we have

$$[D_\mu, D_\nu]\psi = iF_{\mu\nu}\psi \tag{68}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \tag{69}$$

which is the tensor version of equation ....(35) $F_{\mu\nu} = iF^a_{\mu\nu}\tau_a$ and $A_\mu = A^a_\mu \tau_a$, we have

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \frac{1}{2}\epsilon^a{}_{bc}(A^b_\mu A^c_\nu - A^b_\nu A^c_\mu) \\ &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^a{}_{bc} A^b_\mu A^c_\nu. \end{aligned} \tag{70}$$

Under an infinitesimal gauge transformation, each of the components of ψ change by $\delta\psi^A$ ($A = 1, 2$).

$$\delta\psi^A = \psi^{'A} - \psi^A = \epsilon^a(\tau_a)^A_B \psi^B \tag{71}$$

where ϵ^a (a=1,2,3) are the infinitesimal parameters of the gauge transformations. It can be easily shown that under these transformations A^a_μ and $F^a_{\mu\nu}$ change by

$$\delta A^a_\mu = \epsilon^a{}_{bc}\epsilon^b A^c_\mu + \partial_\mu \epsilon^a, \tag{72}$$

and

$$\delta F^a_{\mu\nu} = \epsilon^a{}_{bc}\epsilon^b F^c_{\mu\nu}. \tag{73}$$

Note that $F^a_{\mu\nu}$ transforms as the adjoint representation of the group $SU(2)$.

A *metric* can be defined on the group manifold of $SU(2)$ according to

$$g_{ab} = g_{ba} = \epsilon^c{}_{ad}\epsilon^d{}_{bc} \tag{74}$$

which is an example of the *Cartan-Killing metric*. Note that the isospin indices a, b, are raised and lowered by g^{ab} and g_{ab} in the same manner that the spacetime indices $\mu, \nu, ...$ are raised and lowered by the Minkowski metric $\eta_{\mu\nu}$ or $\eta^{\mu\nu}$.

The free-field Lagrangian density of the Yang-Mills field is

$$\mathcal{L}_{free} = \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \tag{75}$$

6. Self-duality, instantons, and monopoles

Solutions of Yang-Mills equations which have the important property

$$F = \pm\ {}^*F \tag{76}$$

are called *self-dual* **(+) or** *anti-self-dual* **(-). In such a case, the equation of motion** $D^*F = 0$ **reduces to the Bianchi identity** $DF = 0$**. (Anti)self-dual fields are therefore solutions of the field equations. Among the most interesting solutions of the Yang-Mills equations are instantons and monopoles. The instanton solution of 'tHooft and Polyakov is given by**

$$A_\mu = -\frac{1}{g_o} \frac{\eta_{c\mu\nu} \tau^c x^\nu}{r^2 + r_o^2} \tag{77}$$

where r_o **is a constant which prevents** A **to become singular at** $r = 0$**. Also** $r^2 = \sum_{\mu=1}^4 (x^\mu)^2$**,** $\eta_{abc} = \epsilon_{abc}$ **for** $a, b, c = 1, 2, 3$**,** $\eta_{a4b} = -\delta_{ab}$**,** $\eta_{ab4} = \delta_{ab}$ **, and** $\delta_{a44} = 0$**. At large distances, the solution (80) approaches the asymptotic form**

$$A_\mu \rightarrow -\frac{i}{g_o} g^{-1} \partial_\mu g \tag{78}$$

where $g = (r^2 + \lambda^2)^{-1}(x^4 + i\tau_a x^a)$**. This asymptotic form is a pure gauge for which the curvature 2-form vanishes:**

$$F = DA = dA + A \wedge A = 0 \tag{79}$$

In other words, the potential 1-form is asymptotically of the Maurer-Cartan type. The corresponding gauge function $g(x^\mu)$ **is a mapping from the base space (** R^4 **) into the bundle space or the group space. The group manifold of** $SU(2)$ **is a 3-sphere. As we shall see later, this has important topological implications.**

The right action of a group element $h \in G$ **on the principal bundle is given by**

$$ph = (x, g)h = (x, gh) = p' \tag{80}$$

Note that $\pi(p') = \pi(p) = x$**. The vector field associated with the infinitesimal action of the Lie group on the principal bundle is called the** *fundamental vector field*

$$\hat{\eta}(p) = \frac{d}{dt}(pe^{t\eta}) \tag{81}$$

or

$$\hat{\eta}(x, g) = (x, g\eta) \tag{82}$$

Matter fields may couple to the Yang-Mills field. For example, the V-valued field ϕ transforms as

$$\phi' = g^{-1}\phi \tag{83}$$

where $g \in G$. In other words, ϕ is a mapping from the principal bundle P to the vector space V:

$$\phi : P \to V, \qquad \phi(pg) = g^{-1}\phi(p) \tag{84}$$

V can be the adjoint representation of G. The covariant derivative of ϕ is given by

$$D\phi = d\phi + e[A, \phi] \tag{85}$$

The covariant derivative satisfies the Jacobi relation

$$[D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]] + [D_\lambda, [D_\mu, D_\nu]] = 0 \tag{86}$$

The minimal coupling of the ϕ-field to the Yang-Mills field is implemented in the following Lagrangian density

$$\mathcal{L} = -\frac{1}{2}F \wedge^* F + \frac{1}{2}|D\phi|^2 - U(\phi) \tag{87}$$

where $U(\phi)$ is the potential term for the ϕ-field. The field equation for ϕ reads

$$^*D^*D\phi = -\frac{\partial U}{\partial \phi}, \quad or \quad D_\mu D^\mu \phi^a = -\frac{\partial U}{\partial \phi_a} \tag{88}$$

In terms of components, equation (75) reads

$$D_\mu F^{\mu\nu} = J^\nu. \tag{89}$$

The ϕ-field provides a current density (source) for the Yang-Mills equation

$$J^a_\mu = e[\phi, D_\mu \tau^a \phi] \tag{90}$$

Note that the underlying symmetry of the Lagrangian demands that J_μ is covariantly conserved ($D_\mu J^\mu = 0$), and at the same time the ordinary conservation law

$$\begin{aligned} \partial_\mu J^\mu &= \partial_\mu \left[\partial_\nu F^{\nu\mu} + e[A_\nu, F^{\nu\mu}]\right) + e\partial_\mu([A_\nu, F^{\mu\nu}])] \\ &= \partial_\mu \partial_\nu F^{\mu\nu} + e\partial_\mu([A_\nu, F^{\mu\nu} - F^{\nu\mu}]) = 0 \end{aligned} \tag{91}$$

is also satisfied. The choice

$$U = \frac{\lambda}{4}(|\phi|^2 - \phi_o^2)^2 \tag{92}$$

where λ and ϕ_o are positive constants leads to the *spontaneous breaking* **of the gauge symmetry (** *Higgs mechanism* **). This mechanism is responsible for the formation of massive vector bosons ($m_A^2 = e^2\phi_o^2$) .**

For a static configuration of the Yang-Mills and Higgs fields, the total energy functional becomes

$$E = \int \left[\frac{1}{4} Tr(F_{ij}F^{ij}) + \frac{1}{2}(D_i\phi, D^i\phi) + U(\phi) \right] d^3x \tag{93}$$

In 1974, 'tHooft and Polyakov introduced a static, non-singular solution of the Yang-Mills-Higgs equations with remarkable properties. Consider the following *hedgehog* **ansatz:**

$$\phi^a = G(r)\frac{x^a}{er^2}, \quad and \quad A_i^a = [F(r) - 1]\epsilon_{aij}\frac{x^j}{er^2}, \tag{94}$$

where $F(r)$ and $G(r)$ are unknown functions to be determined by field equations (96) and (97) and $A_o^a = 0$. These field equations lead to the following coupled nonlinear differential equations

$$r^2\frac{d^2F}{dr^2} = FG^2 + F(F^2 - 1) \tag{95}$$

$$r^2\frac{d^2G}{dr^2} = 2F^2G + \lambda e^{-2}G(G^2 - \mu^2e^2r^2) \tag{96}$$

These equations can also be obtained by applying the variational principle to the energy functional (94).

In order to be single valued at $r = 0$, A_i^a and ϕ^a should vanish at the origin. This demands

$$F(r) \to 1, \quad and \quad G(r) \to 0 \quad as \quad r \to 0. \tag{97}$$

For a localized, finite-energy solution, the gauge field must reduce to a pure gauge (i.e. Maurer-Cartan form) at large r. The ϕ-field must assume its vacuum (i.e. $|\phi| \to \phi_o$) far from the origin. Therefore,

$$F(r) \to 0, \quad and \quad G(r) \to \mu er \quad as \quad r \to \infty \tag{98}$$

This behavior guarantees the vanishing of energy density in (94) as $r \to \infty$, since in this limit, $F_{ij} \to 0$, $D\phi \to 0$, and $U \to 0$. In other words, although each component of ϕ depends on x^i, the ϕ-field is covariantly "constant".

Note that the vacuum of the ϕ-field corresponds to $U(\phi) = 0$. This defines a 2-sphere in the ϕ_a-space

$$\phi_a\phi_a = (\phi_1)^2 + (\phi_2)^2 + (\phi_3)^3 = \phi_o^2, \tag{99}$$

which is an example of the *manifold of degenerate vacuua***. The asymptotic form of F_{ij}^a at large r corresponds to a radial, inverse square magnetic field**

$$B^i = Tr(\epsilon^{ijk}F_{jk}\phi) = -\frac{x^i}{3r^3} \tag{100}$$

The full solution to equations (96) and (97) can only be obtained numerically. Note that the $SU(2)$ gauge symmetry is spontaneously broken

to $U(1)$ at large distances from the monopole (so is the corresponding bundle). The Higgs mechanism leads to the formation of two massive vector fields from the original A_μ^a's. The remaining massless gauge field is interpreted as the ordinary electromagnetic field which leads to the identification of the magnetic field (101). According to 't Hooft, the following gauge invariant quantity properly describes the electromagnetic field

$$F_{\mu\nu} = \hat{\phi}_a F_{a\mu\nu} = -\frac{1}{g}\epsilon_{abc}\hat{\phi}_a D_\mu \hat{\phi}_b D_\nu \hat{\phi}_c \tag{101}$$

where $\hat{\phi}_a = \phi_a/|\phi|$. With this identification we have

$${}^*F^{\mu\nu} = \frac{1}{2g}\epsilon^{\mu\nu\alpha\beta}\hat{\phi}_a \partial_\alpha \hat{\phi}_b \partial_\beta \hat{\phi}_c \tag{102}$$

and

$$\partial_\nu \ {}^*F^{\mu\nu} = \frac{4\pi}{g}K^\mu \tag{103}$$

where K^μ is the magnetic current. The total magnetic charge is given by

$$Q_m = \frac{1}{g}\int K^0 d^3x = \frac{n}{g} \tag{104}$$

where n is the topological charge or winding number.

There are exact solutions to the field equations in the $\lambda \to 0$ limit which is known as the *Bogomol'nyi-Prasad-Sommerfield* (BPS) limit (Prasad and Sommerfield, 1975, and Bogomol'nyi, 1976). It can be shown that a self-dual or anti-self-dual field satisfies the Yang-Mills equations. The (anti) self-duality condition ($F = \pm\ {}^*F$) leads to the following first-order equations

$$F = \pm{}^* D\phi \quad or \quad F_{ij} = \pm\epsilon_{ijk}D_k\phi. \tag{105}$$

These equations are known as the *Bogomol'nyi equations* . For the hedgehog ansatz (95), equations (106) become the following first order differential equations for $F(r)$ and $G(r)$

$$r\frac{dG}{dr} = G - (F^2 - 1), \tag{106}$$

$$r\frac{dF}{dr} = -FG. \tag{107}$$

These equations can be solved exactly. The monopole solution satisfying the appropriate boundary conditions reads

$$G(r) = \mu e r \coth(\mu e r) - 1 \quad and \quad F(r) = \frac{\mu e r}{\sinh(\mu e r)}. \tag{108}$$

The reader can easily verify that these solutions satisfy the equations (105) and (106) by using the change of variables $1 + G = \mu e r\psi$ and $F =$

$\mu er\chi$. The Bogomol'nyi equations minimize the energy functional (94), with the following total energy

$$E_{min} = \frac{4\pi\mu|g|}{e}. \tag{109}$$

The (anti)self-dual field configurations are particularly important, since they provide the stationary configurations around which quantum fluctuations can be computed. An important property of these models is that the actions are minimized at values which are proportional to the corresponding topological charges.

If instead of being zero, $A_{a0} = \frac{1}{g}J(r)\frac{x^a}{r^2}$ is assumed, we arrive at the following differential equations

$$r^2\frac{d^2K}{dr^2} = K(K^2 - J^2 + H^2), \tag{110}$$

$$r^2\frac{d^2H}{dr^2} = 2HK^2 + \lambda H\left[\frac{1}{g^2}H^2 - r^2F^2\right], \tag{111}$$

and

$$r^2\frac{d^2J}{dr^2} = 2JK^2, \tag{112}$$

in which the same ansatz for A_{ai} and ϕ_a have been used as before. Solutions to these equations are called *dyons* and they bear both electrical and magnetic charges. The electric charge of a dyon is given by

$$Q_e = -\frac{8\pi}{g}\int\frac{JK^2}{r}dr \tag{113}$$

and is not necessarily quantized. It can be shown that in the BPS limit, $Q_e = \frac{4\pi}{g}\sinh\gamma$ where γ is an arbitrary constant.

7. Topological currents

Perhaps the simplest example of a topological current is the one associated with the sine-Gordon system. Consider the sine-Gordon Lagrangian density in 1+1 dimensions

$$\mathcal{L}_{SG} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - (1 - cos\phi) \tag{114}$$

where ϕ is a real scalar field on a two dimensional spacetime $x^o = t$ and $x^1 = x$ with the metric $\eta_{\mu\nu} = diag(-1, 1)$. The self-interaction potential $V(\phi) = 1 - \cos\phi$ has an infinite number of degenerate vacuua at $\phi_n = 2n\pi$, $n \in Z$. The Lagrangian density (115) leads to the famous sine-Gordon equation

$$\partial^\mu\partial_\mu\phi = sin\phi \tag{115}$$

which is known to be an integrable equation with a hierarchy of multi-soliton solutions (Lamb, 1980, and Riazi and Gharaati, 1998). Localized, finite-energy solutions of (116) satisfy the following boundary conditions

$$\phi(+\infty) = 2n\pi, \quad \phi(-\infty) = 2m\pi, \tag{116}$$

where m and n are integers. Topological current for the sine-Gordon system is given by

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi, \tag{117}$$

where $\epsilon^{\mu\nu}$ is the totally antisymmetric tensor in two dimensions. The current density (116) is conserved

$$\partial_\mu J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\mu\partial_\nu\phi = 0 \tag{118}$$

since $\epsilon^{\mu\nu}$ is antisymmetric while $\partial_\mu\partial_\nu$ is symmetric. The total charge of a localized solution (both static and dynamic) is easily shown to be quantized

$$Q_{SG} = \int_{-\infty}^{+\infty} J^o dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\epsilon^{o1}\partial_1\phi dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\partial\phi}{\partial x}dx = \frac{1}{2\pi}\left[\phi(+\infty) - \phi(-\infty)\right] = n - m \tag{119}$$

in which the boundary conditions (117) are used. Note that solutions with different topological charges belong to distinct topological sectors. They are separated from each other by infinite energy barriers. In other words, they cannot be continuously deformed into each other.

The concept of topological charges can be extended to more complicated fields in higher dimensions. Consider, for example, a time-dependent complex scalar field $\phi = \phi_1 + i\phi_2$ on the complex plane z. The vacuum manifold of ϕ is assumed to reside at

$$|\phi|^2 = \phi_1^2 + \phi_2^2 = \phi_o^2 \tag{120}$$

where ϕ_o is a real, positive constant. The topological current for this field can be defined as

$$J^\mu = \frac{1}{2\pi\phi_o^2}\epsilon^{\mu\nu\alpha}\epsilon_{ab}\partial_\nu\phi_a\partial_\alpha\phi_b \tag{121}$$

where $\mu,\nu,\alpha = 0,1,2$ and $a,b = 1,2$. It can be easily shown that this current is conserved ($\partial_\mu J^\mu = 0$). The total charge of a localized field configuration on the complex plane ($z = x + iy$) is

$$Q = \int\int J^o dxdy = \frac{1}{2\pi\phi_o^2}\int\int\epsilon^{ij}\epsilon_{ab}\partial_i\phi_a\partial_j\phi_b dxdy. \tag{122}$$

This can be rewritten as $\int\int\vec{\nabla}\times\vec{H}.d\vec{S}$ where $d\vec{S} = dxdy\hat{k}$, and $\vec{\nabla}\times\vec{H}.\hat{k} = \epsilon_{3ij}\partial_i H_j$ with

$$H_j = \frac{\epsilon_{ab}\phi_a\partial_j\phi_b}{2\pi\phi_o^2} \tag{123}$$

Note that a z-axis is artificially introduced to facilitate the notations of ordinary vector analysis. $\hat{k}$ is the unit vector in the positive z-direction. Stokes's theorem can now be used

$$Q = \oint_C \vec{H}.d\vec{l} \tag{124}$$

where $d\vec{l}$ is a line element along the boundary curve C which is assumed to be in the form of a circle on the complex plane with radius $r \to \infty$. Parametrizing dl and $d\phi$ along C according to

$$dl = rd\theta, \qquad d\phi = \phi_o d\alpha, \tag{125}$$

leads to

$$\begin{aligned} Q &= \frac{1}{2\pi\phi_o^2}\int_{\theta=0}^{2\pi}\frac{1}{2}(\phi_1\frac{\partial\phi_2}{\partial\theta} - \phi_2\frac{\partial\phi_1}{\partial\theta})rd\theta \\ &= \int_{\theta=0}^{2\pi}\frac{d\alpha}{d\theta}d\theta = \frac{1}{2\pi}\oint_{\theta=0}^{2\pi} d\alpha(\theta) = n \end{aligned} \tag{126}$$

where n is an integer corresponding to the number of times the ϕ field circles its vacuum S^1 as the path C is completed *once* on the complex plane. We saw in section 3 that n was properly called the winding number.

For an isovector field ϕ_a ($a = 1, 2, 3$) with an S^2 vacuum

$$\phi_a\phi_a = \phi_o^2 \tag{127}$$

the topological current can be defined as (Vasheghani and Riazi, 1996)

$$J^\mu = \frac{1}{4\pi\phi_o^3}\epsilon^{\mu\nu\alpha\beta}\epsilon_{abc}\partial_\nu\phi_a\partial_\alpha\phi_b\partial_\beta\phi_c \tag{128}$$

Note that the spacetime is now the ordinary Minkowski spacetime and $\mu, \nu, ... = 0, 1, 2, 3$ with $x^o = t$. Once again, the current is identically conserved ($\partial_\mu J^\mu = 0$), and the total charge is quantized

$$Q = \int J^o d^3x = \frac{1}{4\pi\phi_o^3}\oint\frac{dS_\phi}{dS_x}dS_x = n \tag{129}$$

where dS_ϕ and dS_x are area elements in the x-space and ϕ-space, respectively. The current (129) can be written as the covariant divergence of an anti-symmetric second-rank tensor

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{130}$$

where

$$F^{\mu\nu} = \frac{1}{4\pi\phi_o^3}\epsilon^{\mu\nu\alpha\beta}\left[\epsilon_{abc}\phi_a\partial_\alpha\phi_b\partial_\beta\phi_c + \partial_\beta\mathcal{B}_\alpha\right] \tag{131}$$

in which $\mathcal{B}_\alpha$ is an auxiliary vector field. It is interesting to note that the dual field ${}^*F$ with the following tensorial components

$${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = 2\epsilon_{abc}\phi_a\partial^\mu\phi_b\partial^\nu\phi_c + \partial^\mu\mathcal{B}^\nu - \partial^\nu\mathcal{B}^\mu \tag{132}$$

satisfies the equation

$$\partial_\mu\ {}^*F^{\mu\nu} = 0 \tag{133}$$

provided that the vector field $\mathcal{B}^\mu$ is a solution of the following wave equation

$$\Box\mathcal{B}^\mu - \partial^\mu(\partial_\alpha\mathcal{B}^\alpha) = 2\epsilon_{abc}\partial_\alpha(\phi_a\partial^\mu\phi_b\partial^\alpha\phi_c) \tag{134}$$

The right hand side of this equation defines another conserved current

$$K^\mu = 2\epsilon_{abc}\partial_\alpha(\phi_a\partial^\mu\phi_b\partial^\alpha\phi_c) \tag{135}$$

which is consistent with the vanishing of the divergence of the left hand side of equation (135). The resemblance of equations (131) and (134) to Maxwell's equations and the capability of this model to provide non-singular models of charged particles is discussed in Vasheghani and Riazi (1996). Let us write (132) in the following form

$$G = F - H \tag{136}$$

where

$$G^{\mu\nu} = \frac{1}{4\pi\phi_o^3}\epsilon^{\alpha\beta\mu\nu}\epsilon_{abc}\phi_a\partial_\alpha\phi_b\partial_\beta\phi_c \tag{137}$$

and

$$H^{\mu\nu} = \frac{1}{4\pi\phi_o^3}\epsilon^{\mu\nu\alpha\beta}\partial_\beta\mathcal{B}_\alpha \tag{138}$$

We now have

$$dF = 0,$$

and

$$d^*H = 0. \tag{139}$$

Any 2-form like G which can be written as the sum of two parts satisfying (140) is said to admit *Hodge decomposition* **. We shall see in the next section that forms like G and H are cohomologous (i.e. they belong to the same cohomology class), since they differ only by an exact form.**

Topological currents of defects in various space dimensions can be formulated in a unified way. Duan et al. (1999) considered the following topological current for point defects in a medium represented by an $n-$dimensional order parameter ϕ^a ($a = 1, ..., n$):

$$J^\mu = \frac{1}{A(S^{n-1})(n-1)!}\epsilon^{\mu\mu_1\ldots\mu_n}\epsilon_{\hat{\phi}_1\ldots\hat{\phi}_n}\partial_{\mu_1}\hat{\phi}^{a_1}...\partial_{\mu_n}\hat{\phi}^{a_n} \tag{140}$$

where $A(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the area of the $(n-1)$-dimensional unit sphere, and

$$\hat{\phi}^a = \frac{\phi^a}{|\phi|}. \tag{141}$$

The number of space dimensions is also assumed to be n. The topological charge density $\rho = J^o$ which represents the defect density, is everywhere zero except at the location of the defects where it diverges. It therefore behaves like a delta function, and can be represented as

$$J^\mu = \delta(\phi)D^\mu(\frac{\phi}{x}) \tag{142}$$

where $\delta(\phi)$ is the Dirac delta function, and the $n-$dimensional Jacobian determinant is defined by

$$\epsilon^{a_1...a_n} D^{\mu}(\frac{\phi}{x}) = \epsilon^{\mu\mu_1...\mu_n} \partial_{\mu_1}\phi^{a_1}...\partial_{\mu_n}\phi^{a_n}. \tag{143}$$

The defects are therefore located at points where the equations

$$\phi^a = 0 \tag{144}$$

are satisfied. The defect density is given by (Liu and Mazenko, 1997):

$$\rho = J^o = \delta(\phi)D^o(\frac{\phi}{x}) \tag{145}$$

where $D^o = |\frac{\partial(\phi^1...\phi^n)}{\partial(x^1,...,x^n)}|$ is the ordinary Jacobian determinant. The total topological charge is given by

$$Q = \int J^o d^n x = \sum_{i=1}^{l} \beta_i \eta_i \tag{146}$$

where β_i and η_i are the Hopf indices and Brouwer degrees of the ϕ mapping, respectively, and l is the number of point defects in the system.

8. Cohomology and electromagnetism

The theory of cohomology groups was developed by G. de Rham in 1930's. In this section we present a brief introduction to the subject. The relevance of cohomological methods to other gauge fields is discussed in (Henneaux, 1988).

Closed $p-$forms on a manifold $\mathcal{M}$ form a vector space denoted by $C^p(\mathcal{M})$. These forms are also called *p-cocycles*. The subspace of $C^p(\mathcal{M})$ which comprises exact $p-$forms is denoted by $B^p(\mathcal{M})$. Exact and closed $p-$forms are called *p-coboundaries* . The p-th cohomology group is defined as the quotient group $C^p(\mathcal{M})/B^p(\mathcal{M})$:

$$H^p(\mathcal{M}) = C^p(\mathcal{M})/B^p(\mathcal{M}) \tag{147}$$

The elements of $H^p(\mathcal{M})$ form equivalence classes which are called *p-th cohomology classes*.

Two closed $p-$forms ω and η are *cohomologous* (i.e. belong to the same cohomology class) if their difference is an exact form

$$\omega - \eta = d\rho \tag{148}$$

where ρ is a $(p-1)-$form. The p-th de Rham cohomology group is the set of these equivalence classes. Note that the 0-cohomology class is the class of exact $p-$forms (in other words, all exact $p-$forms are cohomologous).

Let us list a few well-known results relevant to the cohomology groups (Guillemin and Pollack, 1974)

- **Exact 0-forms do not exist.**
- $H^p(R \times M)$ **is isomorphic to** $H^p(M)$.
- $H^p(R^k) = 0$ **if** $k > 0$ **and** $p > 0$. **Every closed** $p-$**form (** $p > 0$ **) on** R^k **(** $k > 0$ **) is exact.**
- **The previous item can be generalized to** $H^p(\mathcal{M}) = 0$ **for all** $p > 0$ **if** $\mathcal{M}$ **is contractible.**
- $H^p(S^k)$ **is one dimensional for** $p = 0$ **and** $p = k$. **For all other** $k > 0$, $H^p(S^k) = 0$.
- $H^o(\mathcal{M})$ **is the space of constant functions on** $\mathcal{M}$ **and its dimension counts the number of connected pieces of the manifold. (e.g.** $dimH^o(R^n) = 1$ **). Manifolds which have globally trivial coordinates (** *e.g.* R^n **) have trivial de Rham cohomologies (** $H^p(R^n) = 0$ **for** $p > 0$ **).**
- **The** *Euler-Poincare' characteristic* **of the manifold** $\mathcal{M}$ **is defined by**

$$\chi_{\mathcal{M}} = \sum_{p=0}^{n} (-1)^p d_p \tag{149}$$

where $d_p = dimH^p(\mathcal{M})$ **is the dimension of the** $p-$**th homology group (called the** *Betti number* **). For an** $n-$**dimensional sphere** S^n, $H^o(S^n) = H^n(S^n) = R$ **and** $H^p(S^n) = 0$ **for** $0 < p < n$. **We therefore have** $\chi(S^n) = 0$ **for** $n = odd$ **and** $\chi(S^n) = 2$ **for** $n = even$. **For odd-dimensional manifolds, the Euler characteristic vanishes as we saw in the above example.**

In the absence of magnetic monopoles $dF = 0$ **where** F **is the electromagnetic field 2-form.** F **is therefore a closed 2-form. It is also exact and can be written as** $F = dA$ **where** A **is the electromagnetic potential 1-form. The phase factor** $exp(i \oint_c A)$ **which appears in the quantum mechanical context (e.g. the Aharonov-Bohm effect) can be written as**

$$exp(i \oint_C A) = exp(i \int_S F) \tag{150}$$

using the Stokes's theorem. Here, S **is a surface area with the closed boundary curve** C. **But** $\int_S F$ **is the magnetic flux passing through** S **. The phase factor (151) is known to be observable as shifts in in the interference fringes of electrons in the Aharonov-Bohm experiment. This shows that although** A **is ambiguous up to a** $U(1)$ **gauge transformation, it cannot be assumed redundant at a quantum level. In the presence of magnetic monopoles,** F **is no longer closed, and cannot be derived from a 1-form. A single-valued gauge transformation in the region surrounding a magnetic monopole (over which** F **is closed), leads to a Dirac relation between the electric and magnetic charges (Wu and Yang, 1975)**

$$g = \frac{1}{e} \tag{151}$$

In section 4 we saw that the field of a magnetic monopole can be obtained from the potential 1-form

$$A = -\frac{g}{4\pi} cos\theta d\phi. \tag{152}$$

Although this potential looks smooth over a 2-sphere, the coordinate system itself is singular along the z-axis. This gauge potential, therefore, is not defined on the z-axis. When transformed into Cartesian coordinates, this potential becomes

$$-\frac{g}{4\pi}\left(-\frac{zy}{r(x^2+y^2)}, +\frac{zx}{r(x^2+y^2)}, 0\right)$$

which clearly shows the singularity along the z-axis.

The gauge potential of a magnetic monopole can be chosen as

$$A_{\pm} = \frac{n}{2r}\frac{xdy - ydx}{z \pm r} \tag{153}$$

where $A_{\pm}$ are the potentials over the northern (+) and southern (-) hemispheres of an S^2 centered on the monopole. Over an equatorial strip, $A_{\pm}$ are related by $A_+ = A_- + nd\phi$ which shows that A_+ and A_- do not merge smoothly unless $n = 0$. Note that $\oint_{S^2} F = 2n\pi$ and therefore n represents the quantized magnetic charge of the monopole. The monopole singularity at $r = 0$ should be considered as a hole in the base manifold. The magnetic charge arises from the non-trivial topology of the principal bundle. Occurrence of such integers associated with non-trivial bundles are properly described by the concept of characteristic classes, to be discussed briefly in the next section.

Electric and magnetic charges act as holes in the base manifold (Minkowski spacetime in this case). Outside these holes, F and $^*F$ are closed and their integral over a closed surface are integer multiples of elementary electric and magnetic charges. In fact, even in the absence of magnetic and electric charges, topologically non-trivial curved spaces can lead to similar effects. Consider, for example, the source-free Maxwell's equations on the following so-called *wormhole* spacetime

$$d\tau^2 = dt^2 - dr^2 - (r^2 + r_o^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{154}$$

in which $r_o > 0$ is called the throat radius (Morris and Thorne, 1988). Note that r extends from $-\infty$ to $+\infty$, and the metric (155) describes two asymptotically Minkowskian spacetimes joined by an S^2. Covariant Maxwell's equations $F^{\mu\nu}_{;\mu} = 0$ and $^*F^{\mu\nu}_{;\mu} = 0$ have the following non-singular solutions

$$E(r) = \frac{Q_E}{r^2 + r_o^2}; \qquad B(r) = \frac{Q_M}{r^2 + r_o^2} \tag{155}$$

where E and B are the radial electric and magnetic fields and Q_E and Q_M are constants of integration. The Stokes's theorem $\int_V dF = \int_{\partial V} F$

and a similar relation for $^*F$, when applied to either side of the wormhole imply that the electric and magnetic fluxes through any 2-sphere centered at the wormhole are independent of r. Here, V is a 3-volume confined between two parallel 2-spheres.

9. Homotopy groups and cosmic strings

Let C_1 and C_2 be two loops in a manifold $\mathcal{M}$, based at a point x_o. These two loops are *homotopic* (denoted by $C_1 \simeq C_2$) if they can be continuously deformed into each other. This corresponds to the existence of a continuous set of curves $C(\lambda)$ with $\lambda \in [0,1]$ such that $C(0) = C_1$ and $C(1) = C_2$. Homotopy is an equivalence relation and the set of all closed curves in $\mathcal{M}$ are divided into a set of homotopy classes. The composite curve $C_3 = C_2 \circ C_1$ is defined according to

$$C_3(\lambda) = \begin{cases} C_2(2\lambda) & if \quad 0 \leq \lambda \leq 1/2, \\ C_1(2\lambda - 1), & if \quad 1/2 \leq \lambda \leq 1. \end{cases} \tag{156}$$

Homotopic deformations of any class C provide the identity (Id) of that class. The inverse of a homotopy C^{-1} is defined according to the relation $C^{-1} \circ C = C \circ C^{-1} = Id$. A loop is homotopic to zero if it can be continuously deformed into a point. Such loops are also called *null-homotopic*. Homotopy classes with the inverse and identity defined in this way form a group called the first or *fundamental group* denoted by π_1. A manifold is called *simply connected* if all loops in it are null-homotopic. R^n is an obvious example of a simply connected manifold ($\pi_1(R^n) = 0$). The group manifold of $U(1)$ is a circle (S^1). The fundamental group of $U(1)$ therefore corresponds to the number of times a loop circles the group manifold in the clockwise or counter-clockwise direction. We therefore have $\pi_1(S^1) = Z$, and the same is for $R^2 - \{0\}$. Other examples of fundamental groups include $\pi_1(O(2)) = \pi_1(SO(2)) = \pi_1(U(N)) = Z$, while $\pi_1(SO(3)) = \pi_1(SO(N)) = \pi_1(O(N)) = Z_2$ ($N \geq 3$), where Z_2 is the group of integers modulo 2. Furthermore, $SU(N)$ groups are simply connected and their fundamental group vanishes.

Higher homotopy groups are defined in a similar manner using compact hypersurfaces homotopic to S^n (i.e. continuously deformable to an $n-$dimensional sphere). The $n-$th homotopy group π_n therefore comprises the homotopy classes of maps from S^n to the manifold under consideration. In particular, we have $\pi_n(S^n) = Z$, $\pi_3(S^2) = Z$, and since the group manifold of $SU(2)$ is S^3, $\pi_3(SU(2)) = Z$.

A manifold $\mathcal{M}$ is *p-connected* if all homotopy groups $\pi_i(\mathcal{M})$ vanish for $i \leq p$.

The wormhole space considered in the previous section is simply connected but not 2-connected, since 2-spheres which contain the wormhole cannot be contracted to a point. Homotopy groups have important implications for the existence and stability of topological defects and solitons. Consider, for example, a complex scalar field ϕ with the self-

interaction potential

$$V(\phi) = \frac{\lambda}{4}(\phi^*\phi - \phi_o^2)^2 \tag{157}$$

where λ and ϕ_o are constants. The vacuum manifold is a S^1 on the complex ϕ plane. We can now re-interpret the contents of section 7 (equation 121 onwards) in the framework of homotopy groups. For a localized field on the xy-plane, the $\phi(x,y)$ field along a large circle $x^2+y^2=r^2$ ($r\to\infty$) belongs to a homotopy class mapping the circle $S^1 : x^2+y^2=r^2$ into the circle $S^1 : \phi_1^2+\phi_2^2=\phi_o^2$. This is nothing but the fundamental group S^1: $\pi_1(S^1)=Z$. The topological charge (125) corresponds to the degree of this mapping and labels the corresponding homotopy class. However, the Lagrangian density

$$\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi - \frac{\lambda}{4}(\phi^*\phi - \phi_o^2)^2 \tag{158}$$

does not lead to localized finite energy solutions. This problem can be demonstrated by the $Q=1$ sector with the asymptotic behavior $\phi = \phi_o\exp(i\theta)$ in which $\theta = \arctan(y/x)$. The presence of the $|\nabla\phi|^2$ term in the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}|\nabla\phi|^2 + V(\phi) \tag{159}$$

leads to an energy density

$$u_\phi \simeq |\nabla\phi|^2 = \phi_o^2(\nabla\theta)^2 = \frac{\phi_o^2}{r^2} \tag{160}$$

in plane polar coordinates. This, however, leads to a logarithmically divergent energy integral

$$\int^\infty \frac{\phi_o^2}{r^2} 2\pi r dr = \phi_o^2 \ln r|^\infty \to \infty \tag{161}$$

In order to cure this problem, the global $U(1)$ symmetry of the Lagrangian (159) can be made local

$$\mathcal{L} = D^\mu\phi^* D_\mu\phi - \frac{\lambda}{4}(\phi^*\phi - \phi_o^2)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{162}$$

This leads to the so-called the *Abelian Higgs model*. We demand that the ϕ field be covariantly constant at large r

$$D\phi = 0 \quad \to \vec{\nabla}\phi - ie\vec{A} = 0 \tag{163}$$

with the asymptotic value $\phi = \phi_o\exp(i\theta)$. Equation (164) gives

$$\vec{A} = \vec{\nabla}(-\frac{i\phi}{e}) \quad as \quad r\to\infty. \tag{164}$$

The potential 1-form A is therefore a pure gauge and leads to a vanishing curvature F at large distances. The energy density of the gauge field,

therefore, also vanishes at large distances. Such a solution is non-trivial, since it corresponds to a non-vanishing magnetic flux through the xy-plane

$$\Phi_B = \int F = \int dA = \oint A = \oint \vec{\nabla}(-\frac{i\phi}{e}).\vec{dl} = \frac{1}{e}\phi_o \oint d\theta = \frac{2\pi}{e}\phi_o \tag{165}$$

where the integration is performed over an infinitely large disk. Solutions belonging to other homotopy classes with topological charges (127) have magnetic fluxes $\Phi_B = \frac{2\pi n}{e}\phi_o$. These solutions which are in the form of bundles of magnetic lines of force in a three dimensional space are called *cosmic strings* **in cosmological terminology. They can also represent magnetic flux tubes in the Landau-Ginzburg model of superconductivity.**

For a cosmic string, we have a smooth map from the circle S^1 in the configuration space to the $U(1)$ manifold which was also S^1. The Brouwer degree or winding number of this map corresponds exactly to the integers which label the first homotopy group $\pi_1(S^1)$. In this case, the winding number is given by

$$n = \frac{1}{2\pi\phi_o^2} <^* d\phi|id\phi> . \tag{166}$$

Note that (Felsager, 1983)

$$<^* d\phi|id\phi> = -i\int_{R^2} d\bar{\phi}\wedge d\phi = -i\int_{R^2} d(\bar{\phi}d\phi)$$

$$= \lim_{r_o\to\infty} -i\int_{\rho=\rho_o} \bar{\phi}d\phi = -i\phi_o^2 \lim_{\rho_o\to\infty}\int_{\rho=\rho_o} id\phi = 2n\pi\phi_o^2 \tag{167}$$

in which $\bar{\phi}$ is the complex conjugate of ϕ.

The model described above can also be applied to the magnetic flux tubes which form in superconductors. Electrons in a superconductor form pairs. These so-called *Cooper pairs* **can be described by a complex scalar field ϕ which is called the** *order parameter*. **$|\phi|^2$ represents the density of the Cooper pairs. The energy density of a static configuration is given by**

$$\mathcal{H} = \frac{1}{2}|\vec{\nabla}\phi|^2 + \frac{1}{2}\alpha|\phi|^2 + \frac{1}{4}\beta|\phi|^4 + C \tag{168}$$

where C is a constant and α is a parameter which depends on the temperature via

$$\alpha(T) = a\frac{T - T_c}{T_c} \tag{169}$$

in which a is a positive constant and T_c is the critical temperature. For $T > T_c$, the state of minimum energy density happens at $\phi = 0$ while for $T < T_c$, the minimum converts into a S^1 in the $(Re(\phi), Im(\phi))$ plane

$$|\phi|^2 = -\frac{\alpha}{\beta} > 0 \tag{170}$$

The quantity $\xi(T) \equiv \frac{1}{\sqrt{-\alpha}}$ has the dimensions of length and is called the *coherence length* of the superconductor. In the presence of a magnetic field, we have to implement the covariant derivative

$$D = d - i\frac{g}{\hbar}A \tag{171}$$

where A is the magnetic vector potential, and include the EM energy density in (169).

It can be shown that $\vec{B}$ satisfies

$$\nabla^2\vec{B} + \frac{\alpha q^2}{\beta\hbar^2}\vec{B} = 0 \tag{172}$$

showing that the magnetic field is exponentially damped inside a superconductor. The quantity $\sqrt{-\alpha/\beta}q/\hbar$ is called the *penetration length* of the superconductor. The reader notes that the Abelian Higgs model is mathematically equivalent to the Ginzburg-Landau theory of superconductivity. The following correspondence can be stablished between the parameters of the two models

$$e \leftrightarrow g/\hbar, \quad \lambda \leftrightarrow \beta, \quad and \quad \mu^2 \leftrightarrow -\alpha. \tag{173}$$

It can be visualized that a pair of magnetic monopoles with opposite charges may reside at the end-points of a finite string. The quantization of magnetic charge is consistent with the quantization of the magnetic flux through the string ($\Phi_B = g = \frac{2\pi n}{e}$). Since any increase in the distance between the monopole pair is associated with an equal increase in the length of the string, a linearly increasing potential between the monopoles is implied (Felsager, 1983)

$$V(r) = \Lambda r \tag{174}$$

where Λ is the mass per unit length of the string. This model provides a possible mechanism for the confinement of magnetic monopoles. A similar mechanism was suggested for the quark confinement inside the hadrons (Nambu, 1985).

A rotating relativistic string has the interesting property that its angular momentum J is proportional to its mass squared (M^2). This property is observed in the *Regge trajectories* of the baryons with the same isospin and strangeness. A simpler realization of confinement for solitons with fractional topological charges was introduced by Riazi and Gharaati (1998).

10. Characteristic classes

In dealing with non-trivial fiber bundles, transition functions and integrals of the curvature 2-form lead to integers which have a topological origin. Characteristic classes are an efficient and elegant way to distinguish and classify inequivalent fiber bundles.

Consider a $k \times k$ **complex matrix** m**. A polynomial** $P(m)$ **constructed with the components of** m **is a** *characteristic polynomial* **if**

$$P(m) = P(g^{-1}mg) \quad for \quad all \quad g \in GL(k, C) \tag{175}$$

where g **represents a complex matrix belonging to general linear transformations. For example,**

$$\det(1+m) = 1 + S_1(\lambda) + ... + S_k(\lambda) \tag{176}$$

is an invariant polynomial constructed with the $i-$**th symmetrical polynomial**

$$S_j(\lambda) = \sum_{i_1<...<i_j} \lambda_{i_1}...\lambda_{i_j} \tag{177}$$

where λ_i**s are the eigenvalues of** m**. Curvature 2-forms** Ω **which are matrix-valued have** $P(\Omega)$ **which are closed and have invariant integrals (Chern, 1967). The** *total Chern form* **is defined as**

$$C(\Omega) = \det(1 + \frac{i}{2\pi}\Omega) = 1 + c_1(\Omega) + c_2(\Omega) + ... \tag{178}$$

where $c_i(\Omega)$ **is a polynomial of degree** i **in the curvature 2-form** Ω**:**

$$c_o = 1; \qquad c_1 = \frac{i}{2\pi} Tr\Omega,$$

$$c_2 = \frac{1}{8\pi^2}\{Tr(\Omega \wedge \Omega) - Tr\Omega \wedge Tr\Omega\}, \quad etc. \tag{179}$$

Note that $c_i = 0$ **for** $2i > n$ **where** n **is the dimension of the base manifold. Closedness causes the Chern forms** $c_i(\Omega)$ **belong to distinct cohomology classes. These classes have integer coefficients. Integrals like**

$$\int_{\mathcal{M}} c_2(\Omega) \quad and \quad \int_{\mathcal{M}} c_1(\Omega) \wedge c_1(\Omega) \tag{180}$$

are invariant integers called *Chern numbers* **.**

The action for (anti)self-dual field configurations is proportional to the second Chern number

$$S = -\frac{1}{2}\int TrF \wedge^* F = \mp\frac{1}{2}\int TrF \wedge F = 4\pi|C_2| \tag{181}$$

where $C_2 = \frac{1}{8\pi}\int TrF \wedge F$ **is the second Chern number. The** $|C_2| = 1$ **case corresponds to the 't Hooft instanton.**

The Chern form is a global form on the base manifold and does not depend on the frame chosen. Chern classes are also closely related to the homotopy theory, since the set of isomorphic classes of $i-$**dimensional vector bundles is isomorphic to the homotopy classes of maps from the base manifold to** $Gr(m, i, C)$ **where** $Gr(m, i, C)$ **is the Grassmann manifold of i-planes in** C^m**.**

As a simple example, consider the $U(1)$ bundle of the Dirac monopole over a S^2. We have

$$det(1 + \frac{i}{2\pi}\Omega) = 1 + \frac{i}{2\pi}\Omega \tag{182}$$

and therefore $c_1 = i\Omega/2\pi$. Note that c_1 is real since Ω is pure imaginary ($\Omega = iF = idA$). Furthermore, using equation (154) we have

$$\int_{S^2} c_1 = -\frac{1}{2\pi}\int_{S^2_+} dA_+ - \frac{1}{2\pi}\int_{S^2_-} dA_- = -\frac{1}{2\pi}\int_o^{2\pi} nd\phi = -n \tag{183}$$

which shows that the Chern number and the monopole charge are essentially the same for this example. Note that the integration over S^2 is divided into two hemispheres S^2_+ and S^2_- and $dA_+ - dA_- = nd\phi$ is used.

The characteristic classes of real vector bundles are called *Pontrjagin classes*. **In similarity with the definition (179), the** *total Pontrjagin class* **of a real $O(k)$ bundle is defined as**

$$P(\Omega) = \det(1 - \frac{1}{2\pi}\Omega) = 1 + p_1 + p_2 + ... \tag{184}$$

where Ω is the bundle curvature. The orthogonality conditions lead to the vanishing of odd-degree polynomials. The invariant polynomials are closed here also and the resulting cohomology classes are independent of the connection form.

One can also attribute Pontrjagin classes to the electromagnetic field in the following way (Eguchi et al., 1980)

$$\det(1 - \frac{1}{2\pi}F) = 1 + p_1 + p_2 \tag{185}$$

where F is the electromagnetic field tensor in its matrix form, and p_1 and p_2 are related to the EM energy density and Poynting vector according to $p_1 = \frac{1}{(2\pi)^2}(E^2 + B^2)$ and $p_2 = \frac{1}{(2\pi)^4}(\vec{E}.\vec{B})^2$.

11. Differential geometry and Riemannian manifolds

We closely follow Eguchi et al. (1980) in this section. The metric on a Riemannian manifold $\mathcal{M}$ can be written in the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^a\eta_{ab}e^b \tag{186}$$

where $e^a = e^a_\mu dx^\mu$ is the vierbein basis of $T^*(\mathcal{M})$ and η_{ab} is the flat metric ($\eta_{ab} = \delta_{ab}$ for a Euclidean manifold). The connection 1-form $\omega^a_b = \omega^a_{b\mu}dx^\mu$ obeys the Cartan structure equations

$$T^a = de^a + \omega^a_b \wedge e^b \tag{187}$$

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b \tag{188}$$

where $T^a = \frac{1}{2}T^a_{bc}e^b \wedge e^c$ is the torsion 2-form and $R^a_b = \frac{1}{2}R^a_{bcd}e^c \wedge e^d$ is the curvature 2-form. The Cartan identities read

$$dT^a + \omega^a_b \wedge T^b = R^a_b \wedge e^b \tag{189}$$

and

$$dR^a_b + \omega^a_c \wedge R^c_b - R^a_c \wedge \omega^c_b = 0 \tag{190}$$

which is nothing but the Bianchi identity.

In tensor components,

$$T^\mu_{\alpha\beta} = \frac{1}{2}(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}) \tag{191}$$

where

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}) \tag{192}$$

are the Christoffel symbols, and

$$R^a_b = \frac{1}{2} R^a_{b\mu\nu} dx^\mu \wedge dx^\nu \tag{193}$$

where

$$R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\nu\beta} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\beta} \tag{194}$$

is the Riemann tensor. As a simple example, consider the 2-sphere

$$ds^2 = r_o^2(d\theta^2 + \sin^2\theta d\phi^2) = (e^1)^2 + (e^2)^2 \tag{195}$$

where r_o is the radius of the sphere, and $e^1 = r_o d\theta$ and $e^2 = r_o sin\theta d\phi$. The structure equations reduce to

$$de^1 = -\omega^1_2 \wedge e^2 = 0 \quad and \quad de^2 = -\omega^2_1 \wedge e^1 = r_o cos d\theta \wedge d\phi \tag{196}$$

where $\omega^1_2 = -cos\theta d\phi$ is the connection 1-form. Note that $T^a = 0$ here. The curvature 2-form becomes

$$R^1_2 = R^1_{212} e^1 \wedge e^2 \tag{197}$$

Note also that

$$R^1_2 = d\omega^1_2 = \frac{1}{r^2} e^1 \wedge e^2. \tag{198}$$

The volume form on S^2 is $\Omega = r_o^2 \sin\theta d\theta \wedge d\phi$ with $\int \Omega = 4\pi r_o^2$.

Metrics can also be defined on the group manifolds like those of $SU(N)$. A metric on G is defined by the inner product

$$(g'(0), h'(0)) = -Tr(g_o^{-1} g'(0) g_o^{-1} h'(0)) \tag{199}$$

where $g(t)$ and $h(t)$ are two curves in G which have an intersection at $t = 0$:

$$g(0) = h(0) = g_o \tag{200}$$

This metric is positive definite and multiplication on right or left correspond to isometries of the metric.

Betti numbers (b_m) can also be ascribed to the Riemannian manifolds. Compact orientable manifolds obey the *Poincare' duality* which states that H^p is dual to H^{n-p}. This implies $b_p = b_{n-p}$ which for a four-dimensional manifold becomes $b_o = b_4$ and $b_1 = b_3$. Note that $b_0 = b_4$

counts the number of disjoint pieces of $\mathcal{M}$, and $b_1 = b_3$ vanishes if the manifold is simply connected. Recall that $\chi = b_o - b_1 + b_2 - b_3 + b_4$ is the Euler characteristic of the manifold. The following is an important theorem which relates the local curvature and the global characteristics of hypersurfaces in R^n:

Gauss-Bonnet Theorem: **For any compact, even-dimensional hypersurface S in R^{n+1},**

$$\int_S \kappa = \frac{1}{2}\gamma_n \chi(S) \tag{201}$$

where κ is the curvature of S (see below), $\chi(S)$ is the Euler characteristic of S and γ_n is the volume of the unit $n-$sphere given by

$$V_n = \frac{2(\pi)^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \tag{202}$$

where Γ is the gamma function ($\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1) = 1$, and $\Gamma(x+1) = x\Gamma(x)$).

In a curved manifold, the Hodge * operation involves the metric determinant g and $\epsilon_{\mu\nu\ldots} = g\epsilon^{\mu\nu\ldots}$:

$${}^*(dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}) = \frac{\sqrt{|g|}}{(n-p)!}\epsilon^{\mu_1\cdots\mu_p}_{\mu_{p+1}\cdots\mu_n} dx^{\mu_{p+1}} \wedge ... \wedge dx^{\mu_n} \tag{203}$$

For a scalar field ϕ on an $n-$dimensional Riemannian manifold, ${}^*\phi = \sqrt{|g|}\phi dx^1 \wedge ... \wedge dx^n$. We also have $\epsilon = {}^*1$, ${}^{**}T = (-1)^{k(n-k)}T$ for Euclidean signature and ${}^{**}T = -(-1)^{k(n-k)}T$ for Minkowskian signature.

Torus $T^2 = S^1 \times S^1$ is a compact 2-dimensional manifold which can be covered by the coordinates $0 \leq (\theta_1, \theta_2) \leq 2\pi$. A torus can not be covered with coordinates which are globally smooth. For example, the θ_1 and θ_2 coordinates are discontinuous at the identified circles $\theta_i = 0$ and $\theta_i = 2\pi$. The 1-forms $d\theta_1$ and $d\theta_2$ are therefore not exact. These two 1-forms provide a basis for the first cohomology group of T^2, with dim$H^1(T^2) = 2$. Also $H^2(T^2)$ has the basis $d\theta_1 \wedge d\theta_2$ and $dimH^2(T^2) = 1$. We therefore have

$$\sum_{p=0}^{2}(-1)^p dimH^p(T^2) = +1 - 2 + 1 = 0 \tag{204}$$

which is equal to the Euler characteristic of the torus.

Consider an oriented $n-$dimensional hyper-surface X in R^{n+1}. The outward pointing normal $\hat{n}(x)$ to this hyper-surface maps X into an n-sphere S^n and is called the *Gauss map***:**

$$g : X \to S^n \tag{205}$$

The Jacobian of this map $J_g(x)$ is called the curvature of X at x and is denoted by $\kappa(x)$. For an n-sphere, $\kappa = \frac{1}{r^n}$ independent of the point x on S^n.

12. Two dimensional ferromagnet

Consider a three-component scalar field ϕ_a (a=1,2,3) with the following $O(3)$ Lagrangian density (Belavin and Polyakov 1975, Rajaraman 1988)

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_a \partial_\mu \phi_a. \tag{206}$$

The three components of the scalar field are constrained to the surface of a sphere :

$$\phi_a \phi_a = \phi_1^2 + \phi_2^2 + \phi_3^2 = 1. \tag{207}$$

Using the method of Lagrangian multipliers, the corresponding field equation is found to be

$$\Box \phi_a - (\phi_b \Box \phi_b)\phi_a = 0. \tag{208}$$

Static configurations on the xy-plane are described by the following equation

$$\nabla^2 \phi_a - (\phi_b \nabla^2 \phi_b)\phi_a = 0 \tag{209}$$

Note that

$$\begin{aligned} \phi_a \phi_a = 1 \quad &\rightarrow \quad \phi_a \vec{\nabla} \phi_a = 0 \\ \rightarrow \quad \nabla.(\phi_a \vec{\nabla}\phi_a) &= \vec{\nabla}\phi_a.\vec{\nabla}\phi_a + \phi_a \nabla^2 \phi_a = 0 \\ \rightarrow \quad \phi_a \nabla^2 \phi_a &= -\vec{\nabla}\phi_a.\vec{\nabla}\phi_a \end{aligned} \tag{210}$$

The total energy of the system is given by

$$E = \int \frac{1}{2} \vec{\nabla}\phi_a.\vec{\nabla}\phi_a d^2x. \tag{211}$$

We therefore have a S^2 fiber sitting on every point of the xy-plane, forming a bundle space which is locally $S^2 \times R^2$. The classical vacuum of the system is at $\phi_a = constant$. E vanishes for the vacuum. The $O(3)$ symmetry is spontaneously broken by the vacuum, which can be arbitrarily chosen to be at $(0,0,1)$. A finite-energy, localized solution of (210) is described by two functions $\phi_1(x,y)$ and $\phi_2(x,y)$ with $\phi_{1,2} \rightarrow 0$ as $r \rightarrow \infty$. The points on the xy-plane can therefore be transformed to a sphere using a stereographich map, in which all points on a circle with $r \rightarrow \infty$ are mapped to the "north pole". This identification is allowed so long as the single-valuedness of the ϕ_a-field is concerned, since all points on this large circle reside at the same vacuum point $(0,0,1)$. In solid state theory, the ϕ_a-field may describe the order parameter of a 2D ferromagnet. The $S^2 \rightarrow S^2$ mapping from the configuration S^2 to the field S^2, therefore allows the following geometrical description of the system:

- **Spin waves are sections of the bundle space which is now compactified to $S^2 \times S^2$.**

- **Spin waves belong to distinct homotopy classes. The appropriate homotopy group is $\pi_2(S^2) = Z$.**
- **Each mapping can be characterized by a winding number n which counts the Brouwer degree of the map.**

This winding number is obtained as

$$Q = \frac{1}{8\pi} \int \epsilon_{ij}\epsilon_{abc}\phi_a \partial_i \phi_b \partial_j \phi_c d^2x$$

$$= \frac{1}{8\pi} \int \epsilon_{ij}\epsilon_{abc}\phi_a \frac{\partial \phi_b}{\partial \theta_l} \frac{\partial \theta_l}{\partial x^i} \frac{\partial \phi_c}{\partial \theta_m} \frac{\partial \theta_m}{\partial x^j} d^2x = \frac{1}{8\pi} \int \epsilon_{lm}\epsilon_{abc}\phi_a \frac{\partial \phi_b}{\partial \theta_l} \frac{\partial \phi_c}{\partial \theta_m} d^2\theta \qquad (212)$$

But $\frac{1}{2}\epsilon_{lm}\epsilon_{abc}\frac{\partial \phi_b}{\partial \theta_l}\frac{\partial \phi_c}{\partial \theta_m} d^2\theta$ is the surface element of the internal S^2. Therefore

$$Q = \frac{1}{4\pi} \int d\vec{S}_{int}.\vec{\phi} = n \qquad (213)$$

Note that $d\vec{S}_{int}$ is parallel to the radius vector $\vec{\phi}$ of the unit sphere in the (ϕ_1, ϕ_2, ϕ_3) space.

This interesting system can also be formulated on the complex plane, using the stereographic projection

$$\omega = \frac{2\phi_1}{1-\phi_3} + i\frac{2\phi_2}{1-\phi_3} \equiv \omega_1 + i\omega_2 \qquad (214)$$

The north pole of the ϕ-sphere is projected to $|\phi| \to \infty$. Note that this projection is conformal (i.e. preserves angles).

Consider the self-duality relation

$$^*d\omega = -id\omega \qquad (215)$$

which leads to the equations

$$\frac{\partial \omega_1}{\partial x^1} = \pm \frac{\partial \omega_2}{\partial x^2}, \qquad (216)$$

and

$$\frac{\partial \omega_1}{\partial x^2} = \mp \frac{\partial \omega_2}{\partial x^1}. \qquad (217)$$

Since these are the familiar analyticity conditions, any analytic function $\omega(z)$ or $\omega(z^*)$, is a solution of (216). It can be shown that such solutions in fact minimize the energy functional

$$E = \int \frac{1}{2}\partial_i\phi_a\partial_i\phi_a d^2x = \int \frac{|d\omega/dz|}{(1+\frac{1}{4}|\omega|^2)^2} \qquad (218)$$

The topological charge for these solutions is

$$Q = \pm\frac{1}{4\pi}E. \qquad (219)$$

A simple solution with

$$Q = \frac{1}{4\pi} E = n \tag{220}$$

is given by

$$\omega(z) = \frac{(z - z_o)^n}{\lambda^n} \tag{221}$$

where z_o and λ are constants. Note that Q and E do not depend on λ and z_o which shows that the solutions can be scaled up or down or displaced on the z-plane.

It is seen that the nonlinear $O(3)$ model is quite simple, yet rich in structure. This simple model provides a good insight into the more complicated systems like the Yang-Mills instantons.

13. Instantons in the CP_N model

Real projective spaces $P_N(R)$ are lines in R^{N+1} which pass through the origin. For example, $P_3(R) = SO(3)$. CP_N is the complex version of P_N.

Consider $N+1$ complex scalar fields ϕ_a ($a = 1, 2, ..., N+1$), with the following Lagrangian density (for more details, see Eichenherr 1978)

$$\mathcal{L} = \partial^\mu \phi_a^* \partial_\mu \phi_a + \phi_a^* (\partial^\mu \phi_a) \phi_b^* \partial_\mu \phi_b \tag{222}$$

in which summation over μ and a indices is implied. x^μ ($\mu = 1, 2$) is considered to be the (xy) Euclidean plane. The complex fields are subject to the constraint

$$|\phi_1|^2 + ... + |\phi_{N+1}|^2 = 1 \tag{223}$$

or $\phi_a^* \phi_a = 1$. These complex fields form an N-dimensional complex projective (CP_N) space. It is interesting to note that the Lagrangian density (223) is invariant under the local gauge transformation

$$\phi_a'(x) = \phi_a(x) e^{i\Lambda(x)} \tag{224}$$

without a need to introduce any gauge potentials. A vector field $A^\mu(x)$, however, can be defined according to

$$A^\mu = i\phi_a^* \partial^\mu \phi_a \tag{225}$$

which is real since $\phi_a^* \partial^\mu \phi_a$ is pure imaginary. Using this definition, the Lagrangian density (223) can be written as

$$\mathcal{L} = (D^\mu \phi_a)^* D_\mu \phi_a \tag{226}$$

where

$$D_\mu \phi_a = (\partial_\mu + iA_\mu)\phi_a \tag{227}$$

in resemblance to the covariant differentiation in electromagnetism. Note, however, that A^μ is not a new degree of freedom, and it is determined

solely in terms of the ϕ_a field. Localized solutions with finite energy should satisfy the following asymptotic behavior

$$\phi_a \rightarrow C_a e^{i\psi(\theta)} \tag{228}$$

where C_a are constants satisfying $C_a^* C_a = 1$, and $\psi(\theta)$ is the common phase of the fields (θ is the polar angle in the xy-plane). This phase implies the winding number

$$Q = \frac{1}{2\pi} \int d\psi = \frac{1}{2\pi} \oint \frac{d\psi}{d\theta} d\theta = n \tag{229}$$

where n is an integer. Similar to what we had in the case of cosmic strings, a $\pi_1(S^1)$ homotopy group is involved and solutions having different winding numbers belong to distinct homotopy classes.

14. Skyrme model

T.H.R. Skyrme introduced a nonlinear model in 1960s, with soliton solutions approximately describing baryons (Skyrme, 1961). This model is based on the nonlinear sigma model, augmented with a nonlinear term in the Lagrangian which stabilizes the solitons. The Skyrme Lagrangian is therefore given by

$$\mathcal{L}_{Sk} = -\frac{f_\pi^2}{4} Tr(U^\dagger \partial_\mu U U^\dagger \partial^\mu U) + \frac{1}{32\alpha^2} Tr([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2) \tag{230}$$

where U is the field described by a unitary matrix, α is a dimensionless coupling constant ($\simeq 5$), and f_π^2 is the pion decay constant (130-190 MeV). The pion mass term can also be included in (231) by adding the term $-\frac{1}{2} m_\pi Tr(U + U^\dagger)$. Soliton solutions of the Skyrme model are called *Skyrmions*. They have interesting topological properties, in relevance to the low energy properties of baryons. Witten (1983) showed that the Skyrme model is the high-N_c limit of QCD, where N_c is the number of colors. For a review of the mathematical developments in the Skyrme model, the reader is referred to Gisiger and Paranjape (1998). The geometrical aspects of the Skyrme model were first discussed by Manton and Ruback (1986).

Since U becomes constant at spatial infinity, the points in R^3 can be mapped onto a S^3. Soliton solutions, therefore, define a mapping from this S^3 to the group manifold of $SU(2)$ which is also an S^3. These solutions can therefore be classified according to the third homotopy group π_3:

$$\pi_3 : S^3 \longrightarrow S^3 \tag{231}$$

This homotopy group is isomorphic to the group of integers under addition, and the topological charges label the distinct sectors. The topological current of the Skyrme model is given by

$$B^\mu = \frac{1}{24\pi^2} \epsilon^{\mu\nu\alpha\beta} Tr(U^\dagger \partial_\nu U U^\dagger \partial_\alpha U U^\dagger \partial_\beta U) \tag{232}$$

with the corresponding charge identified with the baryon number:

$$B = \frac{1}{24\pi^2}\int d^3x\epsilon^{ijk}Tr(U^\dagger\partial_i UU^\dagger\partial_j UU^\dagger\partial_k U). \tag{233}$$

The simplest Skyrmion is obtained using the ansatz

$$U = e^{i\hat{r}.\vec{\tau}f(r)} \tag{234}$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices, $\hat{r}$ is the unit vector in the radial direction, and $f(r)$ is a function to be determined by the field equation. The boundary conditions which are required to have a well-defined solution are

$$f(0) = \pi \qquad and \qquad f(\infty) = 0. \tag{235}$$

The simplest description of the Skyrme model is in terms of unitary 2×2 matrices U which belong to $SU(2)$.

Houghton et al. (1998) introduced a new ansatz for the Skyrme model

$$U(r,z) = exp(if(r)\hat{n}_R.\vec{\sigma}) \tag{236}$$

where

$$\hat{n}_R = \frac{1}{1+|R|^2}(2Re(R), 2Im(R), 1-|R|^2) \tag{237}$$

Note that the complex coordinate z is related to the polar coordinates θ and ϕ via $z = tan(\theta/2)exp(i\phi)$ and $R(z)$ is a rational map $R(z) = p(z)/q(z)$, where p and q are polynomials of maximum degree N. The boundary conditions $f(0) = k\pi$ ($k \in Z$) and $f(\infty) = 0$ are implied. The degree of the rational map $R(z)$ determines the baryon number ($B = Nk$). Therefore, all baryon numbers can be obtained with $k = 1$, using the ansatz (237). The $N = 1$ case is the Skyrme's original hedgehog ansatz. The ansatz (237) leads to the following expression for the total energy (Houghton et al. 1998)

$$E = \int \left[\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + (\lambda_1\lambda_2)^2 + (\lambda_2\lambda_3)^2 + (\lambda_1\lambda_3)^2\right] d^3x \tag{238}$$

where λ_1^2, λ_2^2, and λ_3^2 are the eigenvalues of the symmetric strain tensor

$$D_{ij} = -\frac{1}{2}Tr((\partial_i UU^{-1})(\partial_j UU^{-1})). \tag{239}$$

The baryon number density is given by

$$b = \frac{1}{2\pi^2}\lambda_1\lambda_2\lambda_3. \tag{240}$$

It can be shown that

$$\lambda_1 = -f'(r), \quad and \quad \lambda_2 = \lambda_3 = \frac{\sin f}{r}\frac{1+|z|^2}{1+|R|^2}|\frac{dR}{dz}| \tag{241}$$

A Bogomol'nye-type lower limit to the energy functional exists:

$$E \geq 4\pi^2(2N + \sqrt{I}) \tag{242}$$

where

$$I = \frac{1}{4\pi} \int \left(\frac{1+|z|^2}{1+|R|^2} |\frac{dR}{dz}| \right)^4 \frac{2idzd\bar{z}}{(1+|z|^2)^2} \tag{243}$$

The energy limit can be written as $E \geq 12\pi^2 N$, since I itself satisfies the inequality $I \geq N^2$.

15. Solitons and noncommutative geometry

Noncommutative geometry provides a nice tentative framework for the unified description of gauge fields and nonlinear scalar fields responsible for the spontaneous breakdown of gauge symmetries (Connes 1985, Coquereaux et al. 1991, Chamseddine et al., 1993a, Madore 1995, and Okummura et al. 1995).

It has been shown that noncommutative geometry can also deal with gravity and leads to generalized theories of gravity like scalar-tensor theories (Chamseddine et al. 1993b). Here, we briefly describe the general mathematical structure of the Z_2-graded noncommutative geometry and its relevance to the localized soliton-like solutions, closely following Teo and Ting (1997). The reader is referred to this paper for further details.

The Yang-Mills-Higgs theory is formulated using differential forms on $\mathcal{M} \times Z_2$, where $\mathcal{M}$ is an $n-$dimensional Euclidean space and Z_2 is the cyclic group of order two

$$Z_2 = \{e, r | r^2 = e\} \tag{244}$$

An explicit matrix representation of this group is

$$\pi(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad and \qquad \pi(r) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{245}$$

The algebra of complex functions on Z_2 is denoted by $\mathcal{M}_2^+$ and is a subalgebra of $\mathcal{M}_2$ which is the algebra generated by the Pauli matrices. An element of $\mathcal{M}_2^+ \otimes C$ is thus in the following form

$$\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \tag{246}$$

where f_1 and f_2 are complex functions.

A generalized $p-$form on $\mathcal{M} \times Z_2$ looks like

$$\eta = \begin{pmatrix} A + B & C \\ C' & A' + B' \end{pmatrix} \tag{247}$$

where A and A' are $p-$forms on $\mathcal{M}$ (the horizontal $p-$forms), B and B' are $(p-2)-$forms on $\mathcal{M}$, and C and C' are $(p-1)-$forms on $\mathcal{M}$. Note that

η is a composite $p-$form on $\mathcal{M} \times Z_2$, since a generalized $p-$form can be written as $\eta = a \times A$ where a is a $q-$form on Z_2 (vertical part) and A is a $(p-q)-$form on $\mathcal{M}$ (horizontal part). Even-degree forms on Z_2 are diagonal matrices while those of odd degree are off-diagonal. Operations on differential forms like exterior differentiation and Hodge * operation can be extended to the Z_2 forms.

The generalized connection 1-form on $\mathcal{M} \times Z_2$ has the form

$$\omega = \mathcal{A} + \theta + \phi \tag{248}$$

where $\mathcal{A}$ is the Yang-Mills connection (horizontal part of ω), and $\theta + \phi$ is the vertical part associated with the internal Z_2. The corresponding internal symmetry consists of two global $U(1)$'s for the two elements of Z_2, denoted by $U(Z_2)$. Under $g \in U(Z_2)$, θ is the gauge invariant Maurer-Cartan 1-form. The curvature 2-form corresponding to the connection (249) is

$$\Omega = d\omega = i\omega \wedge \omega = F + D_H\phi + m^2 - \phi^2 \tag{249}$$

where $F = d_H\mathcal{A} + i\mathcal{A} \wedge \mathcal{A}$ is the Yang-Mills curvature and

$$D_H\phi = d_H\phi + i\mathcal{A} \wedge \phi. \tag{250}$$

Writing $\mathcal{A}$ and ϕ explicitly in their matrix forms

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad \phi = \begin{pmatrix} 0 & \phi \\ \phi^\dagger & 0 \end{pmatrix} \tag{251}$$

where A and B are ordinary 1-forms and ϕ is a complex scalar field, the curvature 2-form becomes

$$\Omega = \begin{pmatrix} F + m^2 - \phi\phi^\dagger & -D_H\phi \\ -D_H\phi^\dagger & G + m^2 - \phi^\dagger\phi \end{pmatrix} \tag{252}$$

where $F = d_HA + iA \wedge A$, $G = d_HB + iB \wedge B$, and $D_H\phi = d_H\phi + i(A\phi - \phi B)$. The Euclidean action functional then becomes

$$S = \frac{1}{2}\int d^nxTr(\Omega^\dagger_{ij}\Omega^{ij})$$

$$= \int d^nx \left\{ \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}G_{\alpha\beta}G^{\alpha\beta} + 2D_\alpha\phi^\dagger D^\alpha\phi + 2(m^2 - \phi^\dagger\phi)^2 \right\} \tag{253}$$

Note that $D\Omega = 0$ (Bianchi identity) and extremization of (254) leads to $D^*\Omega = 0$ which can be expanded into the following equations

$$D_\beta F_{\alpha\beta} = i(D_\alpha\phi^\dagger\phi - \phi^\dagger D_\alpha\phi),$$

$$D_\beta G_{\alpha\beta} = i(D_\alpha\phi^\dagger\phi - \phi^\dagger D_\alpha\phi^\dagger),$$

and

$$D_\alpha D_\alpha\phi = -2(m^2 - \phi\phi^\dagger)\phi. \tag{254}$$

An element of the composite symmetry group has the form $g = diag(g_1, g_2)$ where g_1 and g_2 belong to the Yang-Mills gauge group. In the special case $g_1 = g_2$ and $A = B$, the action functional reduces to

$$S = \int d^n x \left\{ \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{2} D_\alpha \phi^\dagger D^\alpha \phi + \frac{\lambda}{2} (\phi^\dagger \phi)^2 - \mu^2 \phi^\dagger \phi \right\} \tag{255}$$

where $\mu = em$, $\lambda = e^2$ and the rescaling $\omega \to e\omega$ has been implemented. If the gauge group responsible for the connection A is $SU(2)$, this action will represent the ordinary Euclidean Yang-Mills theory coupled to a Higgs field with $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + ie[A_\alpha, A_\beta]$ and $D_\alpha \phi = \partial_\alpha \phi + ie[A_\alpha, \phi]$. The interesting aspect of the above formalism is that the Higgs field ϕ responsible for the spontaneous symmetry breaking of the Yang-Mills gauge field is now a gauge field associated with the internal Z_2 group. Moreover, the quadratic potential $V(\phi, \phi^\dagger)$ results naturally from the curvature of the connection $\omega = \theta + \phi$ (i.e. $\Omega = d\omega + i\omega \wedge \omega = m^2 - \phi^2$, where only the vertical part has now been considered).

If the manifold $\mathcal{M}$ is one dimensional, the horizontal guage field A dissappears since A_μ has only one component $\mu = 1$ and the corresponding curvature vanishes. The Lagrangian density then reduces to

$$\mathcal{L} = \frac{1}{2} |\frac{d\phi}{dx}|^2 + \frac{e^2}{2} (|\phi|^2 - m^2)^2, \tag{256}$$

which leads to the field equation

$$\frac{d^2\phi}{dx^2} = 2e^2 \phi (|\phi|^2 - m^2). \tag{257}$$

The ansatz $\phi(x) = \chi(x) exp(ikx)$ with real $\chi(x)$, leads to $k = 0$ and

$$\phi(x) = \chi(x) = \pm m \tanh(emx). \tag{258}$$

This is the familiar kink (anti-kink) solution of the nonlinear Klein-Gordon equation in the static case.

The topological current $J = \frac{1}{2m} \frac{\partial \phi}{\partial x}$ leads to the charge $Q = \int J dx = \pm 1$ for the kink (anti-kink). The kink (anti-kink) solution satisfies the generalized (anti-) self-duality relations

$$^*\Omega \wedge \theta = \pm im\Omega \tag{259}$$

This self-duality condition also provides a lower limit for the soliton energy. This lower limit ($E = \frac{4}{3} em^3$) is realized for the kink (anti-kink) solution.

In a 2-dimensional Euclidean space with $U(1)$ as the underlying gauge symmetry (256) reduces to the static abelian Higgs model (see equation 163). The string solution of Nielsen and Olesen fulfills the (anti-) self-duality relation

$$^*\Omega = \pm \Omega \tag{260}$$

similar to that of the non-abelian magnetic monopole. Note that here, $\Omega_{23} = -\frac{1}{m} D_2(Re\phi + iIm\phi\tau_3)$, and $\Omega_{34} = \frac{e}{m^2}(m^2 - \phi^*\phi)\tau_3$ where $B = F_{12}$ is the magnetic field, τ_3 is the third Pauli matrix, and D_α is the $U(1)$ covariant derivative.

In the $n = 3$ case with $G = SU(2)$, the action (256) can be identified with the static Yang-Mills-Higgs system which possesses the monopole solutions of 't Hooft and Polyakov. The (anti) self-duality conditions (261) are now satisfied by the exact solutions of Prasad and Sommerfield in the $\lambda \to 0$ limit.

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